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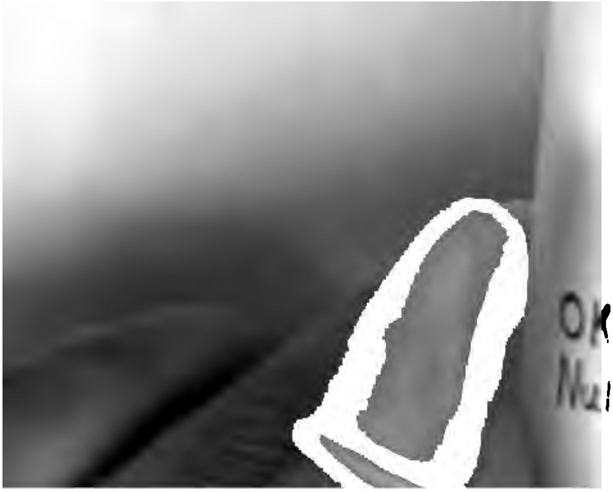

J. M. Schenck

ELEMENTS

OF

GEOMETRY,

THEORETICAL AND PRACTICAL:

ERRATA.

Book I.—Article 17, for (10) read (11). Ar. 18, in the diagrams, let *the lines AB, AB′ terminate in* B, B′. Ar. 33, line 11, for (11) read (12). Ar. 38, for (10) read (11). Ar. 57, line 2, for *proportionally* read *into parts which are proportional to the adjacent sides;* and in the diagram, produce AB *not* DB. Ar. 73, in the second diagram, produce DA *so as to form an exterior angle at A.* Ar. 86, line 9, for *to the* read *to the sides of the.* Ar. 121, line 8, for + *a* read — *a.*

Book II.—Article 15, in the second diagram, cancel the right line BC. Ar. 17, in the diagram, supply *the letter* E; and in line 12, for AC read BC. Ar. 37, line 7, for ABCE read ABCD; and in line 10, for BC.M read AC.M. Ar. 55, third line from the end, for A read C. Ar. 73, in the diagram, supply *the letter* E. Ar. 80, in the diagram, join AB, BC. Ar. 81, line 12, cancel the word *half*; and for ¼ read ½.

Book III.—Article 1, in the diagram supply the letter C. Ar. 53, for (48) read (51).

Book IV.—Article 1, line 5, for *surface* read *solid.* After article 3, supply § 1. Ar. 58, in the diagram, erase the accent over the upper letter A.

PHILADELPHIA:
J. WHETHAM, 22 S. FOURTH STREET.
1836.

Ok

J M Schenck

ELEMENTS

OF

GEOMETRY,

THEORETICAL AND PRACTICAL:

INCLUDING

CONSTRUCTIONS BY THE RIGHT LINE AND BY THE CIRCLE;

TOGETHER WITH THE

MENSURATION OF ALL THE ELEMENTARY PLANE FIGURES AND SOLIDS;

SYSTEMATICALLY ARRANGED,

AND

ADAPTED TO PUBLIC AND PRIVATE INSTRUCTION.

BY EUGENIUS NULTY,

MEMBER OF THE PHILOSOPHICAL SOCIETY, PHILADELPHIA; AND OF THE ACADEMY OF ARTS AND SCIENCES, BOSTON.

Nil nisi certo ordine et certa via sciri posse.—BACON, DIS. OP.

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1836.

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PREFACE.

The present work has been written under the influence of opinions in which the author cannot be regarded as singular. He has presumed that a modern systematical treatise on the Elements of Geometry is essential to modern scientific education, and that any attempt to present those elements with the view of facilitating their more effectual acquirement would meet the approbation of mathematical instructors.

In an elementary course of exact science, the principal requisites are conceived to be a natural sequence of its different parts, and a suitable degree of brevity in their exposition. These have been constantly kept in view in the Treatise now offered to the public; and their observance must greatly contribute to the improvement of students in Geometry, and to the removal of difficulties which have hitherto retarded their progress. With respect to the importance of arrangement, and its effects in the study of this science, the author has had ample experience. He has long observed, that in proportion as the different subjects occupying the attention of students succeeded in natural or in artificial connexion, their ideas appeared distinct or confused, and the impressions left in their minds were of permanent or of momentary continuance.

From these and other considerations relative to demonstrative and inductive science, the author premised in the composition of this work, the following as fundamental principles in regard to the manner and order in which the objects of synthetic Geometry should be introduced and treated.

First, that the possible existence and characteristic property of every object considered, should be established previously to forming its definition.

Secondly, that every object defined should be regarded as establishing the existence of an equal object.

Thirdly, that two or more independent objects should be separately and fully treated; and that the several properties of each should succeed in regular order as to the relative position, proportion, and extension of its corresponding parts.

Fourthly, that different connected objects should follow in natural sequence from the simple to the more complex; and that a like arrangement should be observed in treating two or more objects viewed in connexion.

The necessity of admitting these principles must be obvious to every reader conversant with Geometry, and the advantages which result from them will be easily imagined. Superfluous exertions of memory will undoubtedly be avoided; forced and artificial views will be obviated; an accurate and comprehensive knowledge of the subject will be readily acquired; and the reasoning and inventive powers of the student greatly improved and properly directed. That the present treatise will to the full extent realize the advantages here enumerated, the author has been induced to hope. He believes that the sources in which they originate have seldom escaped his attention; and expects that in case any instances of such unobservance should occur, his general success, if not the acknowledged difficulty of his undertaking, may obtain for him corresponding indulgence.*

The work naturally divides itself into four books; the first two of which involve *plane Geometry;* the last two *solid Geometry,* The principal subjects of those books will be seen in the General Index; and a slight perusal of a few of their subordinate sections will make known, independently of the author's representations, how far simplicity, uniformity, and generality constitute various improvements. As to the equality of the surfaces of triangles established in this treatise without any reference to more complex figures, the reader will consider its novelty and its importance in respect to natural and legitimate arrangement. He will also judge of the necessity of rejecting the usual unsatisfactory demonstrations of incommensurable magnitudes; and will, it is presumed, admit the propriety of adopting, in a work which regards the present state of science, the modern concise and general method of limits, in preference to the indirect and tedious modes of reasoning employed by ancient geometers when treating of the circumferences and areas of circles, and the surfaces and volumes of the different elementary curve or round solids.

---

* The celebrated mathematician D'Alembert was of the opinion that a good work on the Elements of Geometry would not have been unworthy of the powers of Descartes, Newton, Leibnitz, Bernouilli, &c.

To particularize various parts of the work seems hardly necessary. It will be sufficient to remark, that in each book all the chapters but the last, respect the theory and fundamental properties of geometrical magnitudes; and should, by repetition, be rendered perfectly familiar to the student. These last chapters, involving linear and circular constructions, and the mensuration of all the elementary plane figures, and plane and curve solids, will then be readily acquired; and the student thus possessing an accurate and extensive knowledge of theoretical and practical Geometry, will be enabled, with pleasure and advantage, to proceed to higher and more attractive branches of mathematical and mechanical science.

In concluding this preface, the author would mention his partiality to the term *right* applied to *line,** and *equal* applied to *surfaces* and *solids*, as being more analogical and appropriate than the epithets *straight* and *equivalent*, adopted without necessity by some late writers. He would also apprize the reader that he considers proportional magnitudes, when viewed independently of their positions, as forming no part of Geometry; and that he deems all their useful properties as capable of being comprised in a few algebraical theorems. These, with some additional theorems relating to binomial quantities, have been introduced in order to obviate reference to any particular treatise on Algebra, which perhaps may not be in the possession of the student occupied with Geometry.

---

* The term *right line* has been used by distinguished writers on science, as Simpson, Landen, Emerson, Bonnycastle, Hutton, &c., and by Col. Beaufoy, in his extensive work on nautical experiments, now publishing. In Dr Bowditch's finished translation of the Mécanique Céleste, and his important commentary on that work, the term right line is also used; and this gentleman's sanction should establish it in America.

ALGEBRAICAL THEOREMS RELATING TO BINOMIAL AND PROPORTIONAL QUANTITIES.

The demonstrations of the following theorems will be easily understood on a slight acquaintance with algebra, and by the assistance of any work on that science.

---

§ 1. OF THE BINOMIALS $a+b$ and $a-b$.

Theorem 1. . . . $(a+b)+(a-b)=2\,a$.
For this sum is $a+b+a-b$, which is equal to twice the greater quantity a.

Theorem 2. . . . $(a+b)-(a-b)=2\,b$.
For this difference is $a+b-a+b$, which is equal to twice the less quantity b.

Theorem 3. . . . $(a+b).(a-b)=a^2-b^2$.
For this product is $a^2+a\,b-a\,b-b^2$, which is equal to the difference of the squares a^2, b^2.

Remark. The converse is also true; so that the factors of a^2-b^2 are $a+b$ and $a-b$.

Theorem 4. . . $(a\pm b)^2=a^2\pm 2\,a\,b+b^2$.
For the square of $a+b$ is $(a+b)\,(a+b)=a^2+2\,a\,b+b^2$, and the square $(a-b)^2=a^2-2\,a\,b+b^2$; wherefore the two squares $(a\pm b)^2=a^2\pm 2\,a\,b+b^2$.

Remark. The converse is also true; so that $a^2\pm 2\,a\,b+b^2 =(a\pm b)^2$.

---

§ 2. OF THE PROPORTIONAL QUANTITIES a, b, c, d; &c.

Theorem 5. *In any proportion $a : b :: c : d$, the products of the extremes and means constitute the equation $a\,d=b\,c$.*

For this proportion, formed of the equal ratios $a : b$ and $c : d$, constitutes the equation $\frac{a}{b}=\frac{c}{d}$, both members of which being multiplied by $b\,d$, there results $a\,d=b\,c$.

Corollary. If the means b, c be equal, the corresponding proportion $a : b :: b : d$ will give $a\,d=b^2$, in which case b is a *mean proportional* between a and d.

Remark. The proportions $b : a :: d : c$ and $a : c :: b : d$, formed by the *inversion* and *alternation* of similar terms in $a : b :: c : d$, also give $a d = b c$; and conversely from this equation divided by $b d$, or $a c$ and $c d$, there will result the proportion $a : b :: c : d$, or its correlative proportions $b : a :: d : c$ and $a : c :: b : d$.

THEOREM 6. *In any proportion* $a : b :: c : d$, *the sums and differences of corresponding terms, compared to the second and fourth terms, constitute the proportions* $a \pm b : b :: c \pm d : d$.

For $a : b :: c : d$ gives $a d = b c$ (5), each member of which being increased or diminished by $b d$, there results $(a \pm b) d = (c \pm d) b$; whence the proportions $a \pm b : b :: c \pm d : d$ (rem. 5).

Corollary. These proportions give by alternation $a + b : c + d :: b : d$ and $a - b : c - d :: b : d$, which compared to $a : c :: b : d$ and to each other, there will result $a \pm b : a :: c \pm d : c$, and $a + b : a - b :: c + d : c - d$.

Remark. The proportions $a \pm b : b :: c \pm d : d$ are said to be formed by *composition* and *division* respectively from $a : b :: c : d$.

THEOREM 7. *The products of the corresponding terms of the two proportions* $a : b :: c : d$ and $a' : b' :: c' : d'$ *constitute the proportion* $a a' : b b' :: c c' : d d'$.[*]

For these give $a d = b c$ and $a' d' = b' c'$ (5), the corresponding members of which being multiplied together, there results $a a' d d' = b b' c c'$, from which $a a' : b b' :: c c' : d d'$ (rem. 5).

Corollary. If the *antecedent* c of the ratio $c : d$ be equal to the *consequent* d' of $c' : d'$, the compound ratio $c c' : d d'$ becomes simply $c' : d$, so that in this case $a a' : b b' :: c' : d$; and if the *antecedents* a', c' be equal to a, c, and the *consequents* b', d' equal to b, d, there will result the proportion $a^2 : b^2 :: c^2 : d^2$.

Remark. Three or more proportions may be similarly combined; and when the corresponding terms are equal, their cubes or higher powers will be respectively proportional.

THEOREM 8. *The several proportions* $a : b :: a' : b' ::$

---

[*] The quantities a', a'', a''', &c. are read, *a prime*, *a second*, *a third*, &c.

$a'' : b'' ::$ *&c. constitute the proportion* $a : b :: a+a'+a''+$*&c.* $: b+b'+b''+$*&c.*

For the equations $ab=ab$, $ab'=a'b$, $ab''=a''b$, *&c.* (5), give by addition $a(b+b'+b''+\&c.)=b(a+a'+a''+\&c.)$; whence $a : b :: a+a'+a''+\&c. : b+b'+b''+\&c.$ (rem. 5).

GENERAL INDEX.

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ELEMENTS OF GEOMETRY.

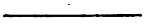
PRELIMINARY PRINCIPLES.

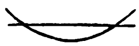
1. The space occupied by any body or solid, when viewed as to its extent, appears enclosed by one or more boundaries called *surfaces*. The particular forms of these surfaces characterize the solid; and the extent of each of them appears limited by one or several boundaries termed *lines*. Each of these lines partakes of the form of the corresponding surface which it limits ; and its extremities appear in determinate positions named *points*. Hence originate the following definitions:

2. A *point* is that which has only position.

3. A *line* is that which has only length.

Corollary. The extremities of a line are points, and the intersections of lines are points.



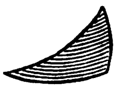


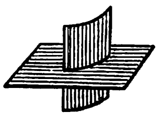
Lines are considered as being *right*, or *curved*.

A *right line* is that which cannot coincide with another right line in any two points without coinciding with it altogether. A *curve line* is that of which no part is a right line

4. A *surface* is that which has only length and breadth.

Corollary. The boundaries of a surface are lines, and the intersections of surfaces are lines.



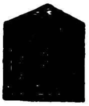


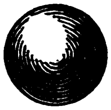
Surfaces are considered as being *plane*, or *curved*.

A *plane*, or plane surface, is that with which a right line cannot coincide in any two points without coinciding with it altogether. A *curve surface* is that of which no part is a plane.

5. A *solid* is that which has length, breadth, and thickness.

Corollary. The boundaries of a solid are surfaces, and the intersections of solids are surfaces.





Solids are considered as being *plane*, or *curved*. A *plane solid* is that which is bounded only by planes. A *curve solid* is partly or entirely bounded by curve surfaces.

6. Geometry is that science which treats of lines, surfaces, and solids. Its elements comprise the simplest of those magnitudes considered synthetically as to their principal relations and properties, and constitute the subject of the present treatise.

BOOK I.

OF RIGHT LINES AND RECTILINEAR FIGURES IN A PLANE.

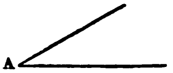
CHAPTER I.

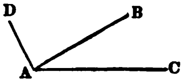
OF THE RELATIVE POSITIONS OF RIGHT LINES.

7. When two right lines have a common point, they must either be in the same direction and constitute a single right line, or diverge from that point and form an opening. In this case, the position of the right lines relatively to each other depends on the opening formed; and this opening is evidently independent of any particular lengths of the diverging lines.

8. *An angle* is the opening formed by two diverging right lines. The point in which the lines meet is called the *vertex*

of the angle formed; and the lines are termed *sides* of this angle.



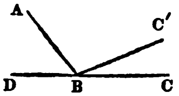


An angle is expressed by a letter placed at its vertex; or by three letters, of which the second is at that point, and the first and third are at the extremities, or in the directions of its sides; as the angle A, the angle B A C or C A B, &c.

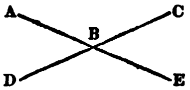
The first mode of expressing an angle may be adopted when there is only one angle at a point; the second becomes necessary when there are two or more angles at a point.

9. Angles take denominations relative to the positions of their sides diverging from the same vertex, or from different vertices.

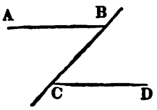
Adjacent angles have a common side and vertex, and have their other sides in the same right line, or forming an angle by their divergence; as A B C, A B D; or A B C′, A B D; or A B C′, C′ B C.



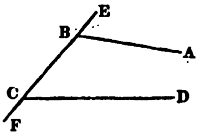
Vertical angles have their sides diverging in opposite directions from a common vertex; as A B C, D B E; or A B D, C B E.



Alternate angles have a side of each in the same right line, relatively to which their other sides diverge in contrary directions; as A B C, B C D.



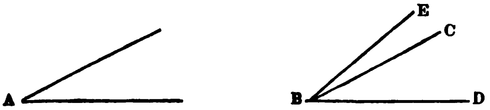
Interior and *exterior* angles have a side of each in the same right line, relatively to which their other sides diverge in similar directions; as A B C, B C D; and A B E, D C F.



Of these angles, A B C is an interior angle, adjacent to the exterior angle A B E; and B C D is an interior angle, opposite to the same exterior angle A B E, on the same side of the right line E F.

10. Angles are either equal or unequal.

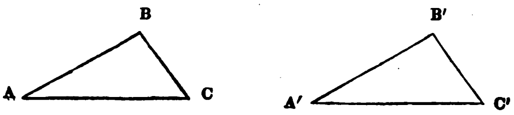
Two *equal* angles have their sides and vertices respectively coincident, when placed on each other. Two *unequal* angles, placed on each other, have two of their sides diverging, when their vertices and the remaining sides are respectively coincident.



Of the preceding angles, A and C B D represent equal angles; A and E B D are unequal angles, the latter being greater than C B D by the angle C B E.

11. Two equal angles may be subtended by *right lines* forming *angles* with their sides; and those *right lines* and *angles* formed, will be respectively *equal*, when the corresponding sides of the equal angles are respectively equal.

There is no reason to conceive those lines and angles unequal; and they evidently become identical, when the vertices and corresponding sides of the equal angles are coincident.

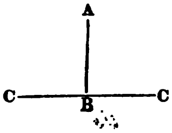


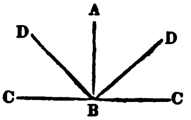
Of the preceding equal angles A, A′, the subtending lines are B C, B′ C′; and the equal angles formed by these equal lines and the corresponding equal sides A B, A′ B′, and A C A′ C′, are respectively B, B′ and C, C′.

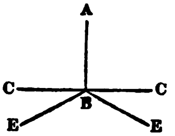
12. Angles are considered as being *right*, or *oblique*, and their sides as being *perpendicular*, or *oblique* to each other.

A *right* angle is that of which either side and the prolongation of the other side, form an equal adjacent angle.

An *oblique* angle is less or greater than a right angle. When less, it is an *acute* angle, when greater, an *obtuse* angle.







Of the preceding angles, A B C is a right angle; A B D an oblique or acute angle, and A B E an oblique or obtuse angle.

Corollary. All right angles are equal to each other; and the corresponding sides of equal and adjacent oblique angles, are not in the same right line, but form angles by their divergence.

13. Either side of a right angle is said to be *perpendicular* to the other side, or to its prolongation; and either side of an oblique angle is said to be *oblique* to the other side, or to its prolongation.

Corollary. There cannot be two perpendiculars to a right line, on the same side, and at the same point of it.

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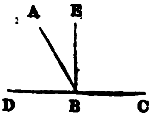
§ 1. OF OBLIQUE AND PERPENDICULAR RIGHT LINES.

THEOREM I.

14. *If a right line meets another, the adjacent angles are together equal to two right angles.*



Let the right line A B meet C D and form the adjacent angles A B C, A B D; these angles are together equal to two right angles.

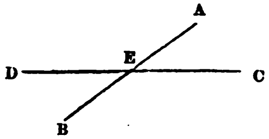
The angles A B C, A B D may be equal or unequal. If they be equal, each of them is a right angle and the theorem evident. If unequal, let the right line B E be perpendicular to C D; the angle A B C is equal to the two angles A B E, E B C, the second of which is a right angle; therefore the two angles A B D, A B C are together equal to the angles A B D, A B E, and this right angle. But the angles A B D, A B E are together equal to the right angle E B D; wherefore the angles A B C, A B D are together equal to two right angles.

15. Two angles, which are together equal to two right angles, are termed *supplemental* angles; and one of these angles is the supplement of the other.

Two adjacent supplemental angles have two of their sides in the same right line; otherwise a third angle supplemental to one of them might be formed unequal to the other, which is impossible.

THEOREM II.

16. *If a right line intersects another, the vertical angles are respectively equal.*

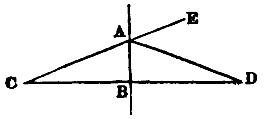


Let the two right lines A B, C D intersect in the point E; the vertical angles A E C, B E D are equal; and also the angles A E D, B E C.

Because the right line A E meets C D, the angle A E C is the supplement of A E D; and because D E meets A B, the angle B E D is also the supplement of A E D; wherefore the angles A E C, B E D are supplements of the same angle, and are equal to each other. In like manner, the vertical angles A E D, B E C are proved to be respectively equal.

THEOREM III.

17. *If a right line intersecting the two sides of an angle be perpendicular to one of them, it is oblique to the other.*



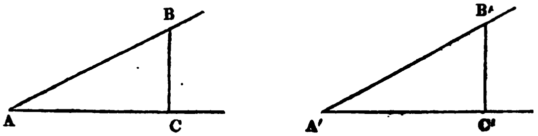
Let the right line A B intersecting the sides C D, C E of the angle C, in the points A, B, be perpendicular to C D; it is oblique to the other side C E.

In C B produced if necessary, let B D be equal to B C, and draw the right line A D. The angle A B D is equal to the adjacent right angle A B C (12); and the sides A B, B D are respectively equal to A B, B C; therefore the angles B A D, B A C, formed by A B and the subtending lines A D, A C, are also equal (10). But these lines meeting the right line C B D, do not form a continued right line C A D; wherefore either of the equal adjacent angles B A C, B A D is an oblique angle, and A B is oblique to the right line C A or C E.

Corollary. From the same point without a right line, there can be drawn only one perpendicular to that line.

18. Two perpendiculars may subtend two equal acute angles, each having a corresponding equal side; and those perpendiculars will be equal, and form equal angles with the corresponding sides.

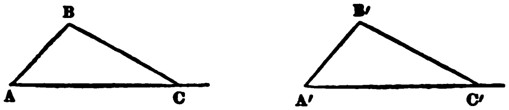
There is no reason to conceive those perpendiculars or angles unequal, and they evidently become identical when the vertices and sides of the equal angles are coincident. (Cor. 17)



Of the equal acute angles A, A′ having the corresponding equal sides A B, A′B′, the equal subtending perpendiculars are B C, B′C′, and the equal angles formed are B, B′.

19. Two oblique lines may subtend two equal angles, each having a corresponding equal side; and those lines, forming equal angles with the corresponding sides, will be equal and form equal angles with the other sides.

There is no reason to conceive those oblique lines or angles unequal; and they become identical when the vertices and sides of the equal angles are coincident.



Of the equal angles A, A′ having the corresponding equal sides A B, A′B′, the equal subtending oblique lines, forming the equal angles B, B′ with these sides, are B C, B′C′; and the equal angles formed by them and the other sides are C, C′.

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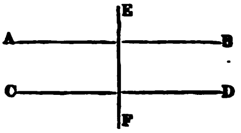
§ 2. OF PARALLEL RIGHT LINES.

20. It is an immediate inference from (17) that two right lines, which are perpendicular to a third right line, cannot meet on either side of this line; for in such case, one of them, meeting the other and forming an angle, would be oblique and not perpendicular to the right line.

21. Two right lines in the same plane are said to be *parallel*, when they cannot meet, however far produced.

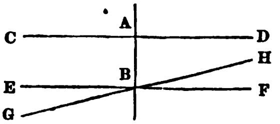
Corollary. Two right lines in the same plane are parallel when they are perpendicular to the same right line.



Of the preceding right lines, A B, C D, regarded as incapable of meeting each other, or as perpendicular to the same right line E F, represent parallel lines.

22. When a right line intersects two parallel lines, and is perpendicular to one of them, it is also perpendicular to the other; or this parallel is oblique to the intersecting line.

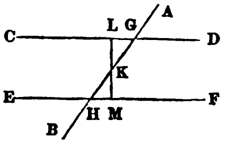
The first asserted position of the right lines constitutes the converse of the preceding corollary; and there appears no reason why this reciprocal perpendicularity and parallelism of the lines should not take place. Were the second position admitted, the oblique parallel and a perpendicular to the intersecting line at the same point would be both parallel to one and the same right line, which must be granted as evidently impossible.



Of the preceding right lines, A B is perpendicular to each of the parallels C D, E F; and B H drawn oblique to A B, and diverging from B F, is considered as capable of meeting the parallel C D.

THEOREM IV.

23. *If a right line intersecting two other right lines makes the alternate angles respectively equal, those two right lines are parallel.*



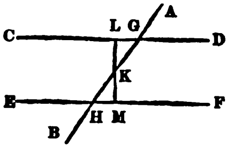
Let the right line A B, intersecting the two right lines C D, E F, make the alternate angles C G H, G H F equal; C D is parallel to E F.

The alternate angles C G H, G H F may be right or oblique angles. If they be right angles, the lines C D, E F will be perpendicular to A B, and will therefore be parallel (cor. 21). But if they be oblique angles, let G K be equal to H K, and let the right line M L, passing through K, be perpendicular to C D. Because the lines G H, L M intersect each other, the angle G K L is equal to H K M (16); and because the subtending lines G L, H M form the equal angles L G K, M H K with the equal sides G K, H K, the angles G L K, H M K are also equal (19). But G L K is a right angle; therefore H M K is also a right angle; and C D, E F being perpendicular to L M, are respectively parallel (cor. 21).

Corollary. If a right line intersecting two other right lines makes the exterior angle equal to the interior and opposite angle on the same side of the line; or makes the two interior angles equal to two right angles; those two lines are parallel. For the alternate angles, which determine the parallelism of the lines, are respectively equal by (16) and (14).

THEOREM V.

24. *If a right line intersects two parallel right lines, the alternate angles are equal.*



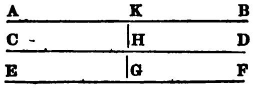
Let the right line A B intersect the two parallels C D, E F; the alternate angles C G H, G H F are equal.

The line A B may be perpendicular or oblique to C D or to E F. If it be perpendicular, the alternate angles C G H, G H F are equal by (22) and (cor. 12). But if it be oblique, let G K be equal to H K, and let M L be perpendicular to C D, E F (22). Because the right line G H intersects L M, the angles G K L, H K M are equal (16); and because these angles have equal sides G K, H K, and are subtended by the perpendiculars G L, H M, the angles L G K, M H K, or the alternate angles C G H, G H F, are respectively equal (18).

Corollary. A right line intersecting two parallel lines makes the exterior angle equal to the interior and opposite angle on the same side of the line; and also makes the two interior angles together equal to two right angles. For either of the alternate angles formed, is equal to the opposite exterior angle; and is also the supplement of the adjacent interior angle (16) and (15).

THEOREM VI.

25. *If two right lines be parallel to a third right line, they are parallel to each other.*

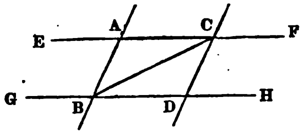


Let the two right lines A B, C D be respectively parallel to the right line E F; A B is parallel to C D.

At any point G in E F, let G H K be perpendicular to this line. Because A B, C D are respectively parallel to E F, they are both perpendicular to G K (22), and are therefore parallel to each other (cor. 21).

THEOREM VII.

26. *If two parallel lines intersect two other parallels, the intercepted parts of those corresponding parallels are respectively equal.*

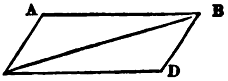


Let the two parallels A B, C D intersect the two parallels E F, G H; the intercepted parts A B, C D, and also A C, B D, are respectively equal.

Join the points B, C. Because the right line B C meets the parallels A C, B D, the alternate angles A C B, C B D are equal (24); and for a like reason, the angles A B C, B C D are equal. But these angles have the side B C common, and are subtended by the corresponding right lines A B, C D and A C, B D; wherefore these lines are respectively equal (19).

THEOREM VIII.

27. *Two equal and parallel right lines are joined by right lines which are also equal and parallel.*



Let A B, C D be two equal and parallel lines joined by A C, B D; these lines are also equal and parallel.

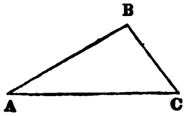
Draw the right line B C. Because B C meets the two parallels A B, C D, the alternate angles A B C, B C D are equal (24); and because these angles have their sides A B, B C and D C, C B respectively equal, the subtending lines A C, B D are also equal, and form the equal angles A C B, C B D (11). But these are alternate angles; wherefore A C, B D are both equal and parallel (23).

CHAPTER II.

OF TRIANGLES.

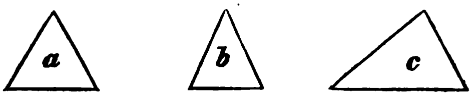
28. When a right line joins the extremities of the sides of any angle, it forms with those sides two other angles; and the three right lines meeting each other, enclose a plane surface, and constitute a *rectilinear* figure, the three angles of which are characterized by the mutual positions of their corresponding sides.

29. A *triangle* is a plane figure contained by three right lines. These lines are its *sides*, one of which is termed the *base;* and the vertex of the opposite angle is considered the *vertex* of the triangle.



Of the triangle A B C, the base is A C, and its vertex is B, that of the angle A B C.

30. A triangle takes different denominations relatively to its sides, all of which may be equal; or two of them may be equal; or the three may be unequal.



An *equilateral* triangle has its sides respectively equal, as (*a*). Its three angles are also equal, since no reason can be assigned to make them unequal.

An *isosceles* triangle has two equal sides, as (*b*). The inclinations of these sides to the base cannot be conceived unlike; and the corresponding angles are respectively equal.

A *scalene* triangle has all its sides unequal, as (*c*). No two of these sides can be said to be equally posited as respects the third side, and the three angles are all unequal.

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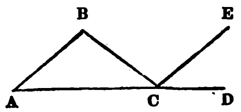
§ 1. OF THE ANGLES AND SIDES OF A TRIANGLE.

THEOREM I.

31. *The three angles of every triangle are together equal to two right angles.*

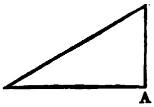


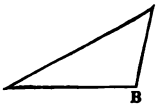
In the triangle A B C, the three angles A, B, B C A, are together equal to two right angles.

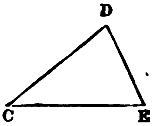
Produce the side A C to D; and at the point C, let the right line C E be parallel to A B. Because the right line A D meets the parallels A B, C E, the exterior angle D C E is equal to the interior opposite angle A; and for a like reason, the angle B C E is equal to the alternate angle B; therefore the whole exterior angle B C D is equal to the two angles A, B. To these add the angle B C A, and the two adjacent angles B C D, B C A are together equal to the three angles A, B, B C A. But these adjacent angles are together equal to two right angles (14); wherefore the angles A, B, B C A are together equal to two right angles.

Corollary. The exterior angle formed by producing any side of a triangle, is equal to the two interior opposite angles, and is therefore greater than either of these angles.

32. A triangle takes denominations relatively to its angles, one of which may be a right angle; or an obtuse angle, the other two angles being acute; or the three angles may all be acute (31).





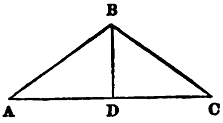


A right angled triangle has one right angle, as A. The side opposite to this angle is called the *hypotenuse;* the two other sides are termed the *base* and *perpendicular.*

An obtuse angled triangle has one obtuse angle, as B; and an *acute angled* triangle has all its angles acute, as C, D, and E. The two last triangles are also denominated *oblique angled* triangles.

THEOREM II.

33. *If two angles of a triangle be equal, the opposite sides are equal; and conversely, if two sides be equal, the opposite angles are equal.*



In the triangle A B C, if the angle A be equal to C, the side B C is equal to A B; and conversely.

At the vertex B, let the right line B D make the angle A B D equal to C B D. Because the angles A, A B D are respectively equal to the angles C, C B D; the exterior angle B D C of the triangle A B D is equal to the exterior angle B D A of the triangle C B D, (cor. 31). But the angles B D C, B D A are right angles, (11), and are subtended by the lines B C, B A, forming with B D the equal angles C B D, A B D; wherefore these lines or sides B C and B A are equal, (19).

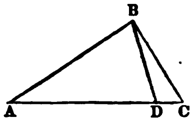
In the converse case, the equal angles A B D, C B D, have the side B D common and B A equal to B C. The equal subtending lines A D, C D, (11), form equal angles with these sides, and the angles A and C are therefore equal.

Corollary. A right line which *bisects* the vertical angle of an *isosceles* triangle, also *bisects* the base and is perpendicular to it, (11.)

Remark. The latter part or converse of this theorem has already been considered as evident, (30). It is here inserted chiefly on account of the corollary resulting from its demonstration.

THEOREM III.

34. *In every triangle, the greater angle is opposite to the greater side; and conversely, the greater side subtends the greater angle.*



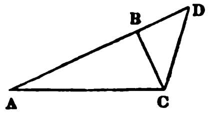
In the triangle A B C, if the side A C be greater than A B, the angle A B C is greater than C; and conversely.

Let A D be equal to A B, and join B D. The angle A D B is the exterior angle of the triangle B D C, and is greater than the interior opposite angle C, (cor. 31). But the angle A D B is equal to A B D, because A B is equal to A D, (33); wherefore the angle A B D is greater than C; and much more is the angle A B C greater than C.

In the converse case, in which the angle A B C is greater than C, the side A C cannot be equal to A B, for then those angles would be equal; nor can A C be less than A B, for then the angle A B C would be less than C; wherefore the side A C must be greater than the side A B.

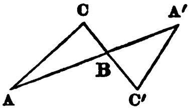
THEOREM IV.

35. *Any two sides of a triangle are together greater than the third side.*



Any two sides A B, B C of the triangle A B C, are together greater than the third side A C.

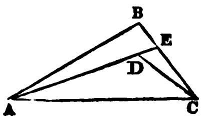
In A B produced, let B D be equal to B C and join C D. Because the sides B C, B D of the triangle B C D are equal, the opposite angles D, B C D are equal, (33); and the angle A C D which is greater than B C D is therefore greater than the angle D. Wherefore A D is greater than A C, (34); but A B, B C are together equal to A D; therefore A B, B C are together greater than A C.



Corollary. Any two right lines which intersect each other, are together greater than two right lines joining their extremities; the four segments at the point of intersection being evidently greater than the two connecting lines.

THEOREM V.

36. *Any two sides of a triangle are together greater than two right lines drawn from any point within the triangle, to the extremities of the third side.*



Any two sides A B, B C of the triangle A B C are together greater than A D, D C, drawn from the interior point D to the extremities of A C.

Produce A D to meet B C in E. The two sides A B, B E of the triangle A B E, are together greater than A E (35); and adding C E, the two sides A B, B C are together greater than A E, E C. For a like reason the sides A E, E C of the triangle A E C are together greater than A D, D C; and therefore, much more are A B, B C together greater than A D, D C.

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§ 2. OF THE ANGLES AND SIDES OF TWO TRIANGLES.

37. The three angles of every triangle, independently of the lengths of its sides, being together equal to two right angles (31), it follows that two triangles may have their angles respectively equal, and their sides unequal. To establish the equality of the angles and sides of any two triangles, they must have at least two corresponding sides equal to each other; and conformably to this restriction, the conditions which determine such equality are comprised in the four following theorems.

THEOREM VI.

38. *If two triangles have two sides, and the included angle of the one respectively equal to two sides and the included angle of the other, the remaining sides and angles are respectively equal.*

This theorem appears to be self-evident (10).

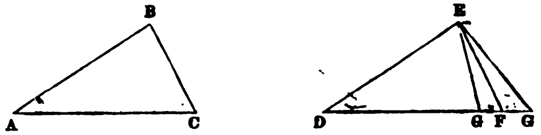
THEOREM VII.

39. *If two triangles have two angles, and the adjacent side of the one respectively equal to two angles and the adjacent side of the other, the remaining angles and sides are respectively equal.*

This theorem appears entirely evident. The triangles obviously become identical when either is applied to the other (19).

THEOREM VIII.

40. *If two triangles have two sides, and an angle of the one respectively equal to two sides and an angle of the other; and if these angles be subtended by corresponding equal sides, the remaining sides and like angles are respectively equal.*



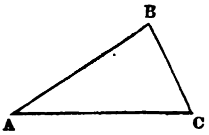
Let A B C, D E F be two triangles of which the sides A B, B C, and the angle A are respectively equal to the sides D E, E F, and the angle D; the sides A C, D F are equal, and the angles B, C are respectively equal to the angles D E F, E F D.

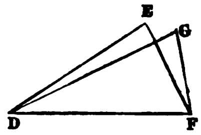
For if the triangle A B C be applied to the triangle D E F, so that the vertices, and sides of the equal angles A, D may be coincident; the side B C will coincide with E F; or if not, it will coincide with E G, a right line terminating in the side D F, or in its prolongation. But the angles E F D, E G D or C being alike acute or obtuse, the supplement of either is obtuse when the other is acute (15), and the contrary; the angles E F G, E G F of the triangle E F G are therefore unequal; and E G or B C is less or greater than the equal side E F (34), which is impossible. The side B C cannot therefore coincide with E G, but must be coincident with E F; and the triangles A B C, D E F being identical, their angles and sides are respectively equal.

THEOREM IX.

41. *If two triangles have the three sides of the one re-*

spectively equal to the three sides of the other, their angles are respectively equal.





Let A B C, D E F be two triangles, of which the sides A B, B C, C A are respectively equal to D E, E F, F D; the angles A, B, C are respectively equal to D, E, F.

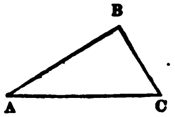
For if the triangle A B C be applied to the triangle D E F, so that the equal bases A C, D F and their corresponding extremities A, D may be coincident, the sides A B, B C will respectively coincide with D E, E F; or if not, they will fall within the triangle D E F, or without it; or one of them as A B, and a contrarily posited side E F of the same triangle, will intersect each other, and form angles subtended by the remaining sides B C, D E. But A B, B C being together equal to D E, E F, can neither fall within nor without the triangle D E F (36); and A B, E F being together equal to D E, B C, cannot intersect and be joined by these lines (cor. 35); wherefore A B, B C must coincide with D E, E F, and the angles A, B, C are respectively equal to the angles D, E, F.

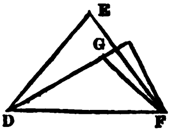
Remark. The truth of this theorem appears evident without demonstration. No reason can be conceived why the equal sides similarly connected should not be similarly posited, and form with each other angles respectively equal in the two triangles; and although the superposition of these triangles does not establish their identity as in other obvious cases, yet the evidence of this identity appears to be necessarily implied in the conditions enunciated.

42. The following theorem and its converse are in some measure allied to the preceding. They respect the inequality of a side and its opposite angle in each of two triangles, the only cases of the kind entitled to particular notice.

THEOREM X.

43. *If two triangles have two sides of the one respectively equal to two sides of the other, but the angles contained by those sides unequal; the third side of that triangle which has the greater angle, is greater than the third side of the other triangle.*





Let A B C, D E F be two triangles in which AB, A C are respectively equal to D E, D F, but the angle E D F greater than the angle A; the side E F is greater than the side B C.

At the point D, let the angle F D G be equal to A, and let the side D G be equal to D E, or A B; join F G. The triangle D G F is equal in all respects to the triangle A B C (38), and G F is therefore equal to B C. The point G may be in the side E F, or it may be within or without the triangle D E F. In the first position, E F is obviously greater than G F or B C; in the second, D E, E F are together greater than D G, G F (36), and D E being equal to D G, E F is greater than G F or B C; and in the third, D G, E F are together greater than D E, G F, and D G, D E being equal, E F is greater than G F or B C.

Corollary. The converse is also true, and the greater angle E D F is opposite to the greater side E F; for this angle cannot be equal to A or less than it, since then the side E F would be equal to or less than B C; conclusions contrary to hypothesis.

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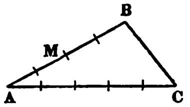
§ 3. OF THE PROPORTION OF THE SIDES OF TRIANGLES.

44. The sides of triangles being right lines of definite and similar extension, may be conceived as formed of equal

parts, and referred to units capable of determining their particular lengths and mutual relations.



Of the triangle A B C, the sides A B and A C are supposed to contain the linear unit M, *four* and *five* times respectively. These sides relatively to this common unit may evidently be represented by the numbers 4 and 5, which determine definitely their particular lengths, and their relation to each other.

45. The sides A B, A C may also be expressed by multiples of 4 and 5, relative to a corresponding submultiple or part of M. Thus the lengths of these lines may be 8 and 10, or 12 and 15, &c., the corresponding units being parts of M contained in this line *twice*, *three times*, &c.

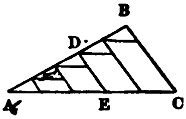
46. The ratio of the sides A B, A C is written A B : A C : and that of their lengths 4 : 5, or 8 : 10, &c. The equality of these ratios constitutes the proportion A B : A C :: 4 : 5, or A B : A C :: 8 : 10, &c.; which is read A B is to A C, as 4 to 5, &c.

47. If either of the sides A B, A C, or both of them, be supposed such as to be incapable of containing *exactly* a common linear unit, their ratio cannot be expressed by numbers, and they are said to be *incommensurable*. With respect to A B, A C thus constituted, an approximating submultiple of either may be taken as a common unit, by which their lengths and corresponding ratio may be numerically determined within any assigned degree of accuracy.

Remark. The proportion of right lines represented by numbers and without regard to their mutual positions, as has been observed at the conclusion of the preface to this work, forms no part of pure geometry, but properly belongs to algebraic investigation. Reference will therefore be made to the general theorems there introduced, whenever the properties of proportional quantities occurring in the course of demonstration, may require particular notice.

THEOREM XI.

48. *If a right line be parallel to one of the sides of a triangle, it divides the other sides proportionally.*



Let A B C be a triangle in which D E is parallel to the side B C, the sides A B, A C are divided proportionally, so that A D : D B :: A E : E C.

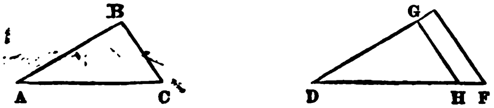
For if A D contains 3 linear units and D B contains 2, making A D : D B :: 3 : 2; and if through the several points of division in A B, right lines be drawn parallel to the sides A C, B C, there will be formed on the equal parts of A B by these parallels, triangles respectively equal (39), and having equal sides parallel to A C, B C. The sides parallel to B C will intercept equal parts of A C (26); and considering one of these parts as a unit, there will result the proportion A E : E C :: 3 : 2. But by hypothesis, A D : D B :: 3 : 2; wherefore by comparison of equal ratios, A D : D B :: A E : E C.

Corollary. A right line parallel to one of the sides of a triangle divides the other sides into parts which are respectively proportional to those sides; for A D, A B and A E, A C have the same ratio 3 : 5; and D B, B A; E C, C A have the ratio 2 : 5.

Remark. The converse of this theorem is also true, since no right lines but those which are parallel to one of the sides can divide the other sides proportionally.

THEOREM XII.

49. *The corresponding sides of two equiangular triangles are respectively proportional.*



Let A B C, D E F be two triangles of which the angles A, B, C are respectively equal to D, E, F; the corresponding sides, opposite to the equal angles, are respectively proportional, as A B : D E :: A C : D F.

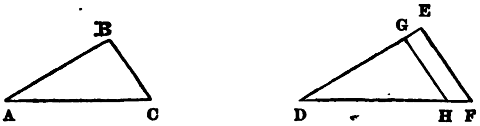
In D E, D F, let D G, D H be respectively equal to A B, A C, and join G H. Because the angles A, D are equal and have equal sides, the angle B is equal to D G H (11); and because the angle B is equal to E, the angle D G H is also equal to E, and G H is parallel to E F (cor. 23). Wherefore D G : D E :: D H : D F; or A B : D E :: A C : D F. In like manner the other corresponding sides are proved to be proportional.

50. *Similar* triangles are those which have their angles respectively equal, and their corresponding or *homologous* sides proportional.

Corollary. Two equiangular triangles are similar (49).

THEOREM XIII.

51. *Two triangles are similar when their sides are respectively proportional.*



Let A B C, D E F be two triangles in which A B : D E :: A C : D F :: B C : E F; the triangles A B C, D E F are similar.

In D E, let D G be equal to A B, and let G H be parallel to E F. The triangles D G H, D E F have a common angle D, and the angles at the subtending parallel sides G H, E F respectively equal (cor. 24); they are therefore similar, and D G : D E :: D H : D F (49). But by hypothesis A B : D E :: A C : D F, and D G is equal to A B; wherefore by comparison of equal ratios, D H is equal to A C. In like manner, G H is proved equal to B C; and the triangles A B C, D G H (41), are equiangular. But the triangles

D G H, D E F are also equiangular; wherefore the triangles A B C, D E F are equiangular and similar.

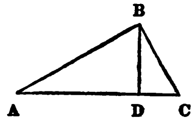
Remark. If the equality of G H, B C be passed without notice, and the last reference changed into (38); this demonstration will prove that;

52. *Two triangles are similar when they have two sides containing an equal angle in each triangle respectively proportional.*

53. The preceding theorems comprise the fundamental properties of similar triangles, the consideration of which, in geometrical researches, occurs perpetually. The following theorems demonstrated by such triangles, will exemplify their application, and sufficiently establish their importance.

THEOREM XIV.

54. *In a right angle triangle, a perpendicular drawn from the right angle to the hypotenuse, is a mean proportional between the segments of this side; either of the sides containing the right angle is a mean proportional between the hypotenuse, and its segment adjacent to this side; and the square of the hypotenuse is equal to the squares of the other two sides.*



Let A B C be a right angled triangle in which B D is perpendicular to the hypotenuse A C; B D is a mean proportional between A D, D C; the sides A B, B C are mean proportionals between A C, A D and A C, C D respectively; and A C^2 is equal to AB^2, B C^2 taken together.

The triangles A B D, D B C have an angle in each common to the triangle A B C, and the angles at D are right angles; these triangles are therefore similar to A B C (31), and to each other; and A D : D B :: D B : D C; or D B is a mean proportional between A D, D C. In like manner,

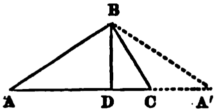
$AC : AB :: AB : AD$ and $CA : CB :: CB : CD$. From these two proportions by multiplying extremes and means, $AC.AD = AB^2$, $AC.DC = BC^2$; the sum of which is $AC.(AD + DC) = AB^2 + BC^2$; or $AC^2 = AB^2 + BC^2$.

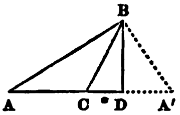
Corollary. The square of either side containing the right angle is equal to the difference of the squares of the hypotenuse and the other side; for $AB^2 = AC^2 - BC^2$; or $BC^2 = AC^2 - AB^2$.

Remark. The last property enunciated in this theorem is entitled to particular consideration. It establishes a numerical connexion between the three sides of a right angled triangle, by means of which any one of them may be determined when the other two are given; and it also leads to an equation involving the sides of any oblique angled triangle, and the segments of either terminated by a perpendicular drawn from the opposite angle. This equation may be thus expressed as a theorem allied to the preceding:

THEOREM XV.

55. *In any oblique angled triangle, the square of the side subtending the oblique angle, is equal to the squares of the two other sides, diminished or increased according as the oblique angle is acute or obtuse, by twice the product of either of these sides, and the right line intercepted between its extremity and a perpendicular drawn to it or to its prolongation from the opposite angle.*





Let ABC be an oblique angled triangle, in which ACB is the oblique angle, and BD a perpendicular to AC or its prolongation; $AB^2 = BC^2 + CA^2 \mp 2\,CA\,.\,CD$.

Because AD is either the difference or the sum of CA and CD; $AD = CA \mp CD$, the square of which is $AD^2 = CA^2 + CD^2 \mp 2\,CA.CD$ (th. 4). Add BD^2 to these equals and $AD^2 + BD^2 = CA^2 + CD^2 + BD^2 \mp 2\,CA.CD$. But AB^2

$=\mathrm{A\,D}^2+\mathrm{B\,D}^2$ and $\mathrm{B\,C}^2=\mathrm{C\,D}^2+\mathrm{B\,D}^2$ (54); wherefore $\mathrm{A\,B}^2=\mathrm{B\,C}^2+\mathrm{C\,A}^2\mp 2\,\mathrm{C\,A.C\,D}$.

Corollary 1. If $\mathrm{D\,A}'=\mathrm{D\,A}$, the triangle A B A′ formed by joining B A′ will be isosceles; $\mathrm{C\,A}'=\mathrm{C\,A}-2\,\mathrm{C\,D}$, and therefore $\mathrm{A\,B}^2=\mathrm{B\,C}^2+\mathrm{C\,A.C\,A}'$, which may be thus expressed:

The square of either of the equal sides of an isosceles triangle, is equal to the square of a right line drawn from the vertical angle to the base, together with the product of the segments of the base.

2. If $\mathrm{C\,A}'=\mathrm{C\,A}$ so that B C may bisect the base of the triangle A B A′, the two triangles A B C, C B A′ will have C B common; and therefore $\mathrm{A\,B}^2+\mathrm{B\,A}'^2=2\,\mathrm{AC}^2+2\,\mathrm{B\,C}^2$; or

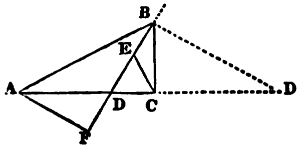
The squares of any two sides of a triangle are together equal to twice the square of half the base, and twice the square of a right line drawn from the vertex to the middle of the base.

Remark. The particular properties just inferred are involved in a general expression connecting the sides of any triangle, a right line drawn from the vertex to the base, and this side, and its segments. For in any triangle A B A′, $\mathrm{A\,B}^2=\mathrm{B\,C}^2+\mathrm{C\,A}^2-2\,\mathrm{C\,A.C\,D}$ and $\mathrm{A'\,B}^2=\mathrm{B\,C}^2+\mathrm{C\,A}'^2+2\,\mathrm{C\,A'.C\,D}$, the sum of which respectively multiplied by C A′, C A gives the expression.

56......$\mathrm{A\,B}^2.\mathrm{C\,A}'+\mathrm{A'\,B}^2.\mathrm{C\,A}-\mathrm{B\,C}^2.\mathrm{A\,A}'=\mathrm{C\,A.C\,A'.A\,A}'$, which may be easily enunciated in ordinary language, if considered necessary.

THEOREM XVI.

57. ***If a right line bisect the vertical angle of a triangle and meet the base, it divides the base proportionally; and the product of the sides containing the bisected angle, is equal to the product of the segments of the base, together with the square of the bisecting line.***



Let A B C be a triangle in which B D bisects the vertical angle; A B : B C :: A D : D C, and A B.B C=A D.D C +B D^2.

At the points C and A, let the angles B C E, D A F be respectively equal to B A D, C B D, and produce B D to meet A F. The triangles A B D, C B E are evidently equiangular, and therefore A B : B C :: A D : C E (49). But the triangle C D E is isosceles, since the angles C D E, C E D are the exterior angles of the preceding triangles; wherefore C E is equal to C D, and A B : B C :: A D : D C.

Again, the triangles A B F, B D C, D A F being evidently similar, A B : B D :: B F : B C, and B D : A D :: D C : D F, the product of the extremes and means of which give A B.B C=B D.B F and B D.D F=A D.D C. But B D.B F =B D.(B D+D F)=B D^2+B D.D F; wherefore by equality A B.B C=A D.D C+B D^2.

Corollary. If A B, B C be respectively equal to A C, A D, there will result A C : A D :: A D : D C, or A D is a mean proportional between the base A C and the segment D C; in which case the right line A C is said to be divided in *extreme and mean ratio*. According to this supposition B D=A D or B C, and the isosceles triangles B A C, C B D have the angles at their bases double their vertical angles.

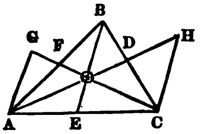
Remark. If the exterior angle at B be bisected by B D′ meeting the base produced in D′, the sides A B, B C will also be proportional to the segments A D′, D′ C; and there will result A D′ : D′ C :: A D : D C; in which case the right line A D′ is said to be divided *harmonically* in the points C and D.

The latter part of this theorem, by virtue of the first part, results from the general expression (56).

THEOREM XVII.

58. *If three right lines be drawn from the angular points of a triangle through any interior point to meet the opposite sides, they will divide those sides into segments, of which*

each three taken alternately round the triangle, form two equal products.



Let A B C be a triangle A E, C D, B F and C E, A F, B D, the alternate segments of its sides, formed by right lines drawn from the angular points through the interior point O, the product A E.C D.B F=C E.A F.B D.

At the points A and C, let A G, C H be parallel to B O E, and produce C F, A D to meet these parallels in G, H. The triangles A O E, A H C are similar; and also the triangles A O G, H O C; wherefore AE : E C :: A G : H C. But the triangles A F G, B F O are similar, and also the triangles C D H, B D O; wherefore A G : B O :: A F : B F and B O : C H :: B D : C D, which multiplied together give the proportion A G : C H :: A F.B D : B F.C D. But A E : E C :: A G : H C, as has been demonstrated; wherefore A E : E C :: A F.B D : B F.C D, the products of the extremes and means of which give A E.C D.B F=C E.A F.B D.

Corollary. When A E=C E; C D.B F=A F.B D and the sides are divided proportionally at D and F.

If A E=C E and also C D=B D, there results A F=B F, so that the bisection of the third side is a consequence of the bisection of the other two sides.

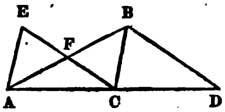
§ 4. OF THE EQUALITY OF THE SURFACES OF TRIANGLES.

59. The conditions mentioned in the section preceding the last, evidently involve the equality of the surfaces of two triangles, as well as that of their angles and sides. But these conditions are not necessary when the equality of two triangles is considered only as respects that of their con

tained surfaces; for this equality of surfaces may evidently subsist, independently of the mutual coincidence of the sides and angles of the triangles.

THEOREM XVIII.

60. *Two triangles which have a common side and vertex, and equal bases in the same right line, are respectively equal.*



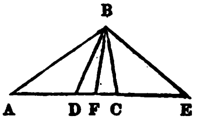
Let A B C, B C D be two triangles, having a common side B C, and equal bases A C, C D in the same right line A D; the triangle A B C is equal to the triangle B C D.

At the point A, let the right line A E be equal and parallel to B C; and join C E intersecting A B in F. Because the angles E A C, B C D are equal (cor. 24), and their contained sides also equal, the triangle E A C is equal to B C D (38); and because E A is equal to B C, and the angles E, E A F respectively equal to F C B, C B F (24), the triangles E A F, F B C are also equal (39). To each of these triangles add A F C, and the triangle E A C is equal to A B C. But the triangle E A C has been proved equal to B C D; wherefore the triangle A B C is equal to B C D.

Corollary. A right line drawn from the vertex of a triangle to the middle of the base, divides the triangle into two equal parts.

THEOREM XIX.

61. *Two triangles which have a common vertex and equal bases in the same right line are equal.*



Let A B C, D B E be two triangles having a common vertex B, and equal bases A C, D E in the same right line A E, the triangle A B C is equal to B D E.

In C D let C F be equal to F D, and join B F. Because A C, D E are equal, and also C F, F D, the remainders A F, F E are equal; and therefore the triangle A B F is equal to F B E (60); and also the triangle D B F is equal to F B C. Add these equal triangles respectively together, and the whole triangle A B C is equal to the corresponding triangle D B E.

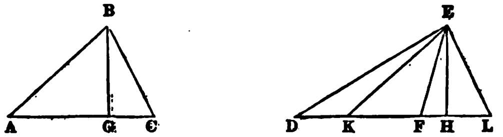
In like manner may the equality of A B C and D B E be proved, should the points C, D be interchanged, by merely reading wholes for remainders, and the contrary.

62. The *altitude* of a triangle is a perpendicular drawn from its vertex to the base, or to the base produced.

Corollary. Two triangles have a common altitude when they have a common vertex, and their bases in the same right line; and triangles having a common altitude and equal bases are respectively equal, according to the preceding demonstration.

THEOREM XX.

63. *Two triangles which have equal bases and altitudes are respectively equal.*



Let A B C, D E F be two triangles of which the equal bases are A C, D F and the equal altitudes B G, E H; the triangle A B C is equal to D E F.

In D H produced, if necessary, let H K, H L be respectively equal to G A, G C; and join E K, E L. The right angled triangles A B G, B G C are respectively equal to K E H, E H L (38); and therefore the triangle A B C is equal to K E L. But the triangle K E L is equal to D E F (61); wherefore the triangle A B C is equal to D E F.

Corollary. Two triangles on equal bases and between the same parallel right lines are equal; for those triangles have equal altitudes respectively perpendicular to the parallels (26).

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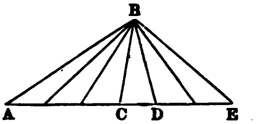
5. OF THE PROPORTION OF THE AREAS OF TRIANGLES.

64. The possibility of dividing triangles into two or more equal parts immediately follows as a consequence from the preceding theorems. These figures may therefore be numerically expressed by referring them to particular units, and their relations may be determined in the same manner as the relations of their sides.

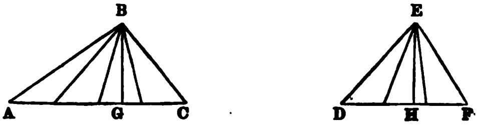


Of the preceding triangles A B C, D B E, the first is supposed to contain three equal parts, and the second two. The ratio of these triangles referred to one of their component parts, considered as a common unit, is therefore the same as the ratio of 3 to 2; and these ratios equated, constitute the proportion A B C : D B E :: 3 : 2, in which for brevity, the word triangle has been omitted before A B C and D B E.

65. The *area of a triangle* is the number which expresses how often it contains a particular unit. The area of A B C may therefore be expressed by 3; that of D B E by 2; and the ratio of these areas is the same as the ratio of their corresponding triangles.

THEOREM XXI.

66. *Two triangles which have equal altitudes are proportional to their bases.*

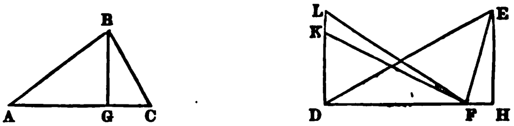


Let A B C, D E F be two triangles of which the altitudes B G, E H are equal; A B C : D E F :: A C : D F.

For if the base A C contains four units of which D F contains three, so that A C : D F :: 4 : 3; and if the vertices and the several points of division in the bases be joined; the triangle A B C will be divided into four equal parts, and the triangle D E F into three; wherefore A B C : D E F :: 4 : 3. But by hypothesis, A C : D F :: 4 : 3; therefore A B C : D E F :: A C : D F.

THEOREM XXII.

67. *Two triangles which have equal bases are proportional to their altitudes.*

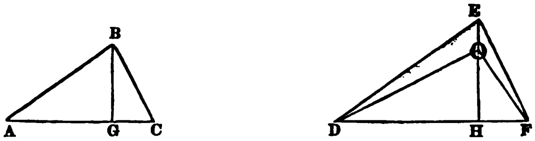


Let A B C, D E F be two triangles of which the equal bases are A C, D F and B G, E H the altitudes; A B C : D E F :: B G : E H.

At the point D, let D L be perpendicular to D F, and let D K, D L be respectively equal to the altitudes B G, E H; join K F, L F. The triangles A B C, D K F have equal bases and altitudes; and are therefore equal (63). For the same reason, the triangles D E F, D L F are equal. But the triangle D K F : D L F :: K D : L D (66); wherefore A B C : D E F :: B G : E H.

THEOREM XXIII.

68. *Two triangles are proportional to the products of their bases and altitudes.*



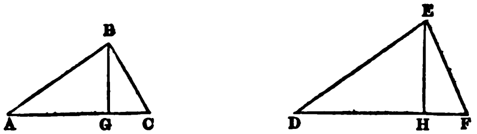
Let A B C, D E F be two triangles of which the bases are

A C, D F, and the altitudes B G, E H; the triangle A B C : D E F :: A C.B G : D F.E H.

In E H, let O H be equal to the altitude B G, and join D O, F O. The triangles A B C, D O F have equal altitudes, and therefore A B C : D O F :: A C : D F (66). The triangles D O F, D E F have the same base, and therefore D O F : D E F :: O H : E H (67). Multiply these equal ratios, and there will result A B C : D E F :: A C.B G : D F.E H (th. 7).

THEOREM XXIV.

69. *Two triangles which have one angle of the one equal to one angle of the other, are proportional to the products of the sides which contain those angles.*

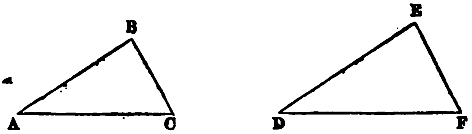


Let A B C, D E F be two triangles in which the angles A, D are equal; A B C : D E F :: A B.A C : D E.D F.

For if A C, D F be the bases, and B G, E H the altitudes, the two right angled triangles A B G, D E H having equal angles A, D are similar, and A B : D E :: B G : E H, which multiplied by the ratio A C : D F gives A B.A C : D E.D F :: A C.B G : D F.E H. But A B C : D E F :: A C.B G : D F.E H (68); wherefore A B C : D E F :: A B.A C : D E.D F.

THEOREM XXV.

70. *Two similar triangles are proportional to the squares of their corresponding sides.*



Let A B C, D E F be two similar triangles in which A, D are equal angles, and AB, DE homologous sides; ABC : DEF :: $AB^2 : DE^2$.

Because the angles A, D are equal, ABC : DEF :: AB.AC : DE.DF; and because the triangles ABC, DEF are similar, AB : DE :: AC : DF, which multiplied by the ratio AB : DE gives $AB^2 : DE^2$ :: AB.AC : DE.DF. Compare this and the first proportion, and ABC : DEF :: $AB^2 : DE^2$.

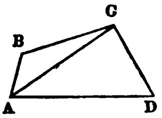
Corollary. Two similar triangles are proportional to the squares of their perimeters; for those perimeters having all their parts respectively proportional, are themselves proportional to any two corresponding parts; and their squares are proportional to the squares of those parts.

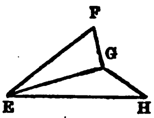
CHAPTER III.

OF QUADRILATERALS.

71. When two right lines subtend any two adjacent angles and diverge at the extremity of the common intermediate side, they form, in connexion with the two other sides, a plane rectilinear figure which has four angles, and which is characterized by the lengths and positions of its bounding sides.

72. A *quadrilateral* is a plane figure contained by four right lines or sides. One of these sides is named the *base*, and the right line which joins the vertices of any two opposite angles is termed a *diagonal*.





The figure ABCD is a quadrilateral, of which AD is the base and AC is a diagonal. The figure EFGH is also a quadrilateral; EH is the base, and EG a diagonal.

In ABCD, the angles have their sides similarly posited as respects the bounded surface. In E F G H this is not the case, and the two sides F G, H G are interiorly directed, and form at G what is technically named a *reentrant angle*, relatively to which the three angles at E, F, H are called *salient angles*. In this treatise no further notice will be taken of reentrant angles, the consideration of which is entirely unnecessary in determining the properties of the sides and areas of rectilinear figures.

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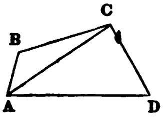
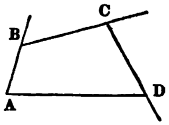
§ 1. OF THE ANGLES AND SIDES OF A QUADRILATERAL.

THEOREM I.

73. *The interior angles of any quadrilateral are together equal to four right angles; and the corresponding exterior angles formed by producing the sides in like directions are also equal to four right angles.*



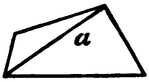


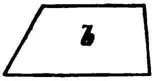
Let A B C D be a quadrilateral; the interior angles A, B, C, D; and also the corresponding exterior angles are equal to four right angles.

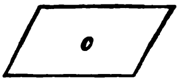
Draw the diagonal A C. The figure A B C D is composed of the two triangles A B C, C D A, and its interior angles A, B, C, D are evidently composed of the angles of these triangles, and which are together equal to four right angles (31); wherefore the interior angles A, B, C, D are equal to four right angles.

In the second case, the exterior and corresponding interior angle at any of the vertices A, B, C, D are together equal to two right angles (cor. 31); therefore all the exterior and interior angles of the figure are together equal to eight right angles. But the interior angles are equal to four right angles; wherefore the exterior corresponding angles are also equal to four right angles.

74. A quadrilateral takes the following denominations relative to the positions and lengths of its sides, which may be unequal or equal:







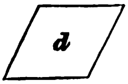
75. A *trapezium* is a quadrilateral of which the opposite sides are not parallel; as figure (*a*). Two of the sides of a trapezium are evidently unequal.

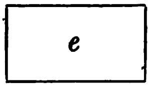
76. A *trapezoid* is a quadrilateral of which only two sides are parallel; as figure (*b*). The two parallel sides of a trapezoid are necessarily unequal.

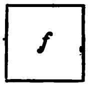
77. A *parallelogram*, or rhomboid, is a quadrilateral of which the opposite sides are parallel; as figure (*c*).

The opposite sides of a parallelogram are respectively equal (27), and also the opposite angles; for a diagonal evidently divides it into two equal triangles.

A parallelogram assumes the following forms:







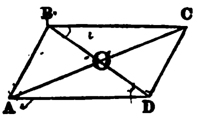
78. A *rhombus*, or lozenge, is a parallelogram of which the sides are all equal; as figure (*d*). The angles of a rhombus are not right angles.

79. A *rectangle*, or oblong, is a parallelogram of which the angles are all right angles; but the sides are not all equal; as figure (*e*).

80. A *square* is a parallelogram of which all the sides are equal, and all the angles right angles; as figure (*f*).

THEOREM II.

81. *The two diagonals of a parallelogram bisect each other.*



In the parallelogram A B C D, the diagonals A C, B D bisect each other in the point O.

Because A C, B D meet the parallel right lines A D, B C, the angles O A D, O D A are respectively equal to O C B, O B C (24); and A D being equal to B C (77), the triangles O A D, O C B have the sides O A, O D respectively equal to O C, O B (39); and therefore A C, B D are bisected at the point O.

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§ 2. OF THE ANGLES AND SIDES OF TWO QUADRILATERALS.

THEOREM III.

82. *Two quadrilaterals which are composed of triangles, having equal angles and sides similarly situated, have their angles and sides respectively equal.*

For the angles of the quadrilaterals are composed of equal angles contained by equal sides, and the figures evidently become identical, when one is applied to the other.

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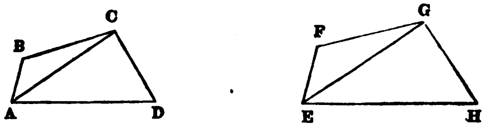
§ 3. OF THE PROPORTION OF THE SIDES OF QUADRILATERALS.

THEOREM IV.

83. *Two quadrilaterals, which are composed of similar triangles similarly situated, are equiangular, and have their homologous sides proportional.*



Let A B C D, E F G H be two quadrilaterals, of which the component triangles A B C, A C D and E F G, E G H, are respectively similar; A B C D, E F G H are equiangular, and A B : E F :: C D : G H :: A D : E H.

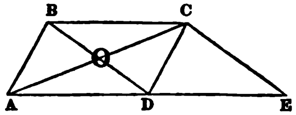
Because the triangles A B C, A C D are similar to E F G, E G H, the whole angles B A D, B C D are respectively equal to F E H, F G H; and the angles B, D being also equal to F, H, the quadrilaterals are equiangular.

Because the triangles A B C, E F G are similar, A B : E F :: A C : E G; and because the triangles A C D, E G H are similar, A C : E G :: C D : G H :: A D : E H. Compare these and the preceding ratios, and A B : E F :: C D : G H :: A D : E H.

Corollary. The perimeters of similar quadrilaterals are proportional to any two homologous sides.

THEOREM V.

84. *The squares of the two diagonals of a parallelogram are equal to the squares of the four sides.*



Let A B C D be a parallelogram; $AC^2+BD^2=AB^2+BC^2+CD^2+DA^2$.

Produce A D; let D E be equal to A D, and join C E. Because A E, the base of the triangle A C E, is bisected, $AC^2+CE^2=2AD^2+2CD^2$ (cor. 55). But C E is equal to B D (77), and A D, C D are respectively equal to the opposite sides B C, A B; wherefore $AC^2+BD^2=AB^2+BC^2+CD^2+DA^2$.

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§ 4. OF THE EQUALITY OF THE SURFACES OF QUADRILATERALS.

THEOREM VI.

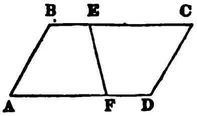
85. *Two parallelograms which have equal bases and altitudes are respectively equal.*

For diagonals divide the parallelograms into triangles which have equal bases and altitudes, and which are respectively equal (63). The parallelograms being composed of these equal triangles, are therefore respectively equal.

Corollary. A parallelogram is equal to a rectangle of the same base, and having the same altitude; and a parallelogram, or a rectangle, is equal to twice a triangle on the same base and between the same parallels.

THEOREM VII.

86. *A parallelogram is equal to twice a trapezoid of the same altitude, and of which the two parallel sides are together equal to the base of the parallelogram.*



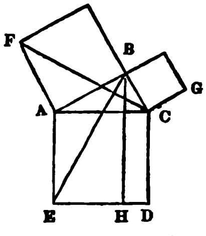
Let A B C D be a parallelogram of which the base A D is equal to the parallel sides A F, B E of the trapezoid A B E F, the parallelogram is equal to twice the trapezoid.

Because A D is equal to B C and F D to B E, A F is equal to E C; but A B is equal to C D, wherefore the trapezoid A B E F has its sides respectively equal to the trapezoid E C D F; and these figures being evidently equiangular, are equal in all respects, and therefore A B C D=2.A B E F.

Corollary. A trapezoid is half a rectangle of the same altitude, and of which the base is equal to the parallel sides of the trapezoid.

THEOREM VIII.

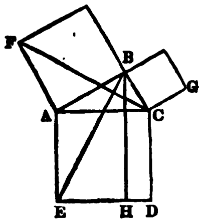
87. *The square on the hypotenuse of a right angled triangle is equal to the two squares on the other sides.*

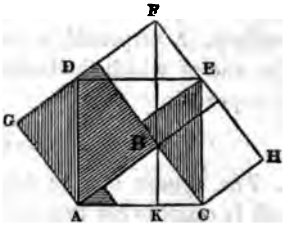


Let A B C be a right angled triangle, of which the hypotenuse is A C; the square A C D E is equal to the two squares B F, B G, on the other sides A B, B C.

Join B E, C F, and let B H be parallel to A E or C D. The angles C A E, B A F are right angles (80), and the angle B A C is adjacent to each; wherefore the angles B A E, C A F are equal. The sides B A, A E and F A, A C of these equal angles are also respectively equal; wherefore the triangle B A E is equal to the triangle F A C (38). But the rectangle A H is twice the triangle B A E (cor. 85); and B C being evidently parallel to A F (cor. 21), the square B F is twice the triangle F A C; wherefore the rectangle A H is equal to the square B F. In like manner, the rectangle C H is proved equal to the square B G; and by addition, the square A C D E is equal to the two squares B F, B G.

Remark. This theorem respecting the equality of the surfaces of squares, must not be confounded with the latter part of (54), in which the squares there mentioned are the second powers of numbers expressing the sides of the triangle, referred to linear units, independently of any considerations connected with (64) or the present section.





Another demonstration.

Let A B C be the right angled triangle, and A C D E the square on the hypotenuse A C. Produce the sides A B, B C parallel to which, and through the points C, D and A, E, draw the right lines C H, F G and A G, F H. Join F B, which produce to meet A C in K. The triangle A B C is evidently equal to D F E, and also to each of the triangles A G D, E H C; wherefore A F, C F are parallelograms, and B G, B H are squares on A B, B C. The parallelogram A F is equal to the rectangle D K, on the same base A D and between the same parallels A D, F K (cor. 63); and is also equal to the square B G; wherefore the rectangle D K is equal to the square B G. For a like reason, the rectangle E K is equal to the square B H; and therefore the whole square A C D E is equal to the two squares B G, B H; or the square of the hypotenuse A C is equal to the squares of the two other sides A B, B C.

Remark. The small shaded triangle at D may be taken from the square B G, and placed at A in the angle B A C. In like manner, the shaded triangle A G D may be removed and placed in the angle B C E. The square B G will thus constitute the shaded part of the square A C D E, which will be completed by the triangle and quadrilateral forming the square B H and separated by the right line C E; and in this mechanical mode, the square on the hypotenuse A C may be ocularly proved equal to the two squares on the sides A B, B C.

§ 5. OF THE PROPORTION OF THE AREAS OF QUADRILATERALS.

THEOREM IX.

88. *Two similar quadrilaterals are proportional to the squares of their homologous sides.*

For those figures are proportional to their similar component triangles, formed by corresponding diagonals; and are therefore proportional to the squares of their homologous sides (70).

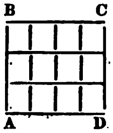
Corollary. Two similar quadrilaterals are proportional to the squares of their perimeters; for these perimeters are evidently proportional to any two of the corresponding sides of which they are composed.

THEOREM X.

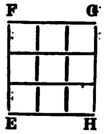
89. *Two rectangles are proportional to the products of their bases and altitudes.*

For diagonals divide those figures into triangles which are respectively proportional to the products of their corresponding bases and altitudes (68). The rectangles composed of these triangles are therefore proportional to the same products of their bases and altitudes.

Remark. By means of this theorem, the areas of quadrilaterals may be determined conformably to the established mode of measuring surfaces. Assume a square of which the side is the linear unit of length, as the unit of surfaces. The area of this square being 1 and the product of its equal base and altitude being also 1; the corresponding area of a rectangle in terms of square units may be determined by the product of the numbers which express its base and altitude; and may be represented by this product.



Thus the area of the rectangle A B C D, of which the base A D and altitude A B contain 4 and 3 linear units, may be represented by the product $4.3=12$, which evidently expresses the number of square units contained in the surface of A B C D.



In like manner, the area of the square E F G H, relatively to the assumed unit, becomes 3.3, or $3^2=9$. This figure may therefore be represented by the second power, or square of its side; and according to such mode of representation, the equality of surfaces forming the subject of (87) becomes a consequence of the important property demonstrated in (54), and noticed in the remark to (87).

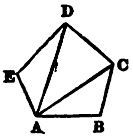
The area of a triangle, or that of a trapezoid, results directly from the area of a corresponding parallelogram, or rectangle; and thus the areas of quadrilaterals may be determined in square units by means of the areas of their component triangles.

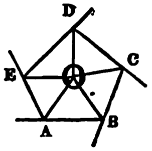
CHAPTER IV.

OF POLYGONS.

90. When three or more angles at a common vertex are subtended by right lines diverging from each other at the extremities of the intermediate sides, these right lines and the two remaining extreme sides, form in successive connexion, a plane rectilinear figure which has five or more angles, and which is characterized by the lengths and positions of its bounding sides.

91. A *polygon* is a plane figure bounded by five or more sides. One of these sides is named the *base;* and a right line joining any two angles not adjacent to the same side, is termed a *diagonal.*





The figure A B C D E is a polygon, of which A B is the base; and A C, A D are diagonals.

§ I. OF THE ANGLES AND SIDES OF A POLYGON.

THEOREM I.

92. *The interior angles of any polygon are together equal to twice as many right angles as the figure has sides less two; and the corresponding exterior angles, formed by*

producing the sides in similar directions, are together equal to four right angles.

For the polygon may be divided by diagonals, drawn from the vertex of any of its angles, into a number of triangles equal to that of its sides less two; and its interior angles being together equal to the angles of these triangles, are therefore equal to twice as many right angles as it has sides less two. With respect to the exterior angles, they are evidently equal to four right angles; since joined to the interior angles, they form a number of right angles equal to twice the number of sides of the polygon.

Instead of dividing the polygon into triangles by diagonals, right lines may be drawn from an interior point to the several vertices of the angles. These right lines will form a number of triangles corresponding to the sides of the polygon; and from the consideration of which, the theorem may also be demonstrated.

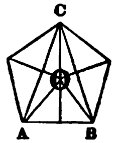
93. A polygon takes the following denominations relative to the number of its sides, as far as twelve inclusive:

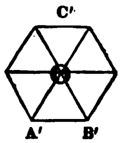
Pentagon, or polygon of five sides; *hexagon*, six sides; *heptagon*, seven sides; *octagon*, eight sides; *nonagon*, nine sides; *decagon*, ten sides; *endecagon*, eleven sides; *dodecagon*, twelve sides.

When the number of sides exceeds twelve, the polygon is designated by that number; as, a polygon of thirteen sides; a polygon of twenty sides; &c.

94. A polygon may have all its angles equal; or all its sides equal; and it may have both its angles and its sides equal.

An *equiangular* polygon has all its angles equal. An *equilateral* polygon has all its sides equal. A *regular* polygon has all its angles and all its sides equal.

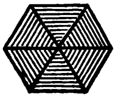


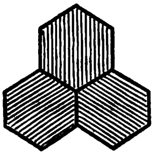


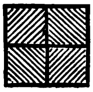
Of the preceding polygons, A B C is a regular pentagon, of which each interior angle is $\frac{3.2}{5}=\frac{6}{5}$ of a right angle (92): A′ B′ C′ is a regular hexagon; each interior angle is $\frac{4.2}{6}=\frac{4}{3}$ of a right angle.

95. A regular polygon may evidently be divided into equal triangles, by right lines diverging from a central point O, which is equally distant from the several vertices of the angles. The number of such equal isosceles triangles being equal to the number of sides, each of the angles at the central point O in the pentagon A B C is $\frac{4}{5}$ of a right angle (92); and joining A C, B C, each of the angles O C A, O C B is $\frac{1}{5}$; so that the isosceles triangle A C B has the angles at its base respectively equal to twice the angle at the vertex C. In the hexagon A′ B′ C′, any of the equal component triangles as O A′ B′ is evidently equilateral, each of the angles being $\frac{2}{3}$ of a right angle; and the radiating lines are equal to the sides of the figure.

It may not be uninteresting to observe that the hexagon, square, and equilateral triangle are the only plane figures which can enclose a surface by their successive repetition. Six equilateral triangles form a hexagon; four contiguous squares form a square; and three hexagons enclose a surface round a point, taken as a vertex common to three adjacent angles.







§ 2. OF THE ANGLES AND SIDES OF TWO POLYGONS.

THEOREM II.

96. *Two polygons which are composed of the same number of triangles, having equal angles and sides similarly situated, have their angles and sides respectively equal.*

For the angles of the polygons are composed of equal angles contained by equal sides, and those figures become identical when one is applied to the other.

§ 3. OF THE PROPORTION OF THE SIDES OF POLYGONS.

THEOREM III.

97. *Two polygons which are composed of similar triangles similarly situated are respectively equiangular, and have their homologous sides proportional.*

The demonstration is precisely similar to that of (83), which may be easily extended to any number of sides by comparing in succession the corresponding triangles which compose the polygons.

Corollary. The perimeters of similar polygons are proportional to any two of their homologous parts.

§ 4. OF THE EQUALITY OF THE SURFACES OF POLYGONS.

98. Two polygons have evidently equal surfaces when their component triangles, whether similarly or otherwise situated, are respectively equal.

THEOREM IV.

99. *A regular polygon is equal to a triangle of which the base is equal to its perimeter, and the altitude equal to that of one of its equal component triangles about a central point.*

For the component isosceles triangles of a regular polygon have equal altitudes and equal bases; and these bases constitute the perimeter of the polygon.

§ 5. OF THE PROPORTION OF THE SURFACES OF POLYGONS.

THEOREM V.

100. *Two similar polygons are proportional to the squares of their homologous sides.*

For their component similar triangles are proportional to the squares of the homologous sides, which form the corresponding boundaries of the polygons.

Corollary. Two similar polygons are proportional to the squares of their perimeters.

THEOREM VI.

101. *The area of a regular polygon, referred to square units, is equal to half the product of its perimeter, and the altitude of one of its equal component triangles, about a central point.*

For the area of a triangle equal to the polygon (99), is half that product (89).

CHAPTER V.

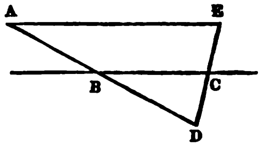
OF THE CONSTRUCTION AND MENSURATION OF RECTILINEAR FIGURES.

102. The subject of this chapter embraces two classes of problems; one resolvable by constructions depending on right lines; the other requiring some elementary knowledge of algebraical formulæ, and their conversion into numbers by the ordinary operations of arithmetic. The former class constitutes a part of what is usually termed *practical geometry*, contradistinguished from this science, theoretically considered, as in the preceding chapters. The latter class involves the determination of the areas of rectilinear figures, and may be viewed as the first and simplest branch of *mensuration*.

§ 1. OF CONSTRUCTIONS RELATIVE TO RIGHT LINES.

PROBLEM I.

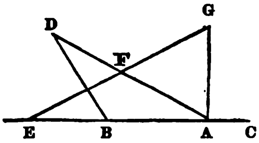
103. *Through a given point* A *to draw a right line parallel to a given right line* B C.

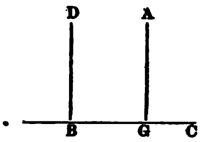


From A draw any right line A D intersecting B C in B, and let B D be equal to A B. From D draw the right line D E intersecting B C in C, and take C E equal to C D. Join A E, which will be parallel to B C. For the sides A D, D E of the triangle A D E are divided equally in B, C; and A E, B C are therefore parallel (rem. 48).

PROBLEM II.

104. *From a given point* A *to draw a perpendicular to a given right line* B C.





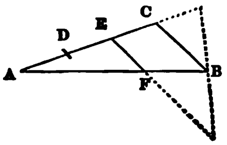
First, let A be in the line B C, and draw any right line B D equal to A B. Join A D, which apply on the given line from A to E, and in A D take A F equal to A B. Draw E F, which produce until F G be equal to E F; join A G, the required perpendicular.

For the triangles A E F, A G F being isosceles, the exterior angles A F G, A F E, which are together equal to two right angles (14), are twice the angles E A F, F A G, constituting the angle E A G, which is therefore a right angle, making A G perpendicular to B C.

Secondly, let A be without B C, and from B draw the perpendicular B D. Through A draw A G parallel to B D (103), which will be perpendicular to B C (22).

PROBLEM III.

105. *To divide a given right line* A B *into two or more equal parts.*

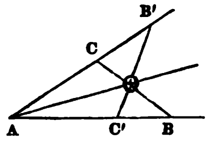


Draw A C, forming with A B an angle B A C; and apply to it any right line A D, as often as there are equal parts required in A B. Join B C, parallel to which and through E, the last division of A C, draw E F, which will intercept B F, one of the required parts of A B.

For the sides A B, A C of the triangle A B C are divided proportionally by E F, and B F, C E are therefore like parts of A B, A C.

PROBLEM IV.

106. *To divide a given angle* A *into two equal parts.*

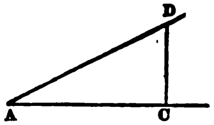


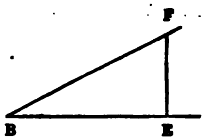
In the sides of the angle A, take A B, A C respectivel equal to A B', A C'; and join B C, B' C', intersecting each other in O. Join A O, which will bisect the angle A.

The triangles A B C, A B' C' are identical, and the angle A C B is equal to A C' B'. The triangles A O C, A O' C' having A O, A C equal to A O, A C', and the angle A C O equal to A C' O are identical (40); and therefore the angle O A C is equal to O A C'.

PROBLEM V.

107. *At a given point* A *to make an angle equal to a given angle* B.



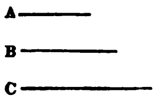


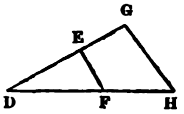
Through A draw the right line A C, which take equal to B E, a side of the angle B. Erect the perpendiculars C D, E F, the latter meeting the other side of B in F. Take C D equal to E F, and draw A D, which will form with A C the required angle.

For the right angled triangles A C D, B E F are evidently identical, and therefore the angle A is equal to B.

PROBLEM VI.

108. *To find a fourth proportional to three given right lines* A, B, C.

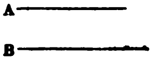


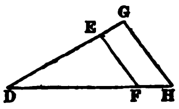


In the sides of any angle D, take D E, D F respectively equal to the lines A, B; and join E F. In D E, produced if necessary, take D G equal to the line C, and through G draw G H parallel to E F and meeting D F produced, in H; the right line D H is the required proportional.

For the right lines E F, G H being parallel, D E : D G :: D F : D H; or, A : C :: B : D H (48).

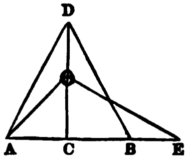
Corollary. When B=C, D G=D F; and in this case, D H is a third proportional to the lines A and B.





PROBLEM VII.

109. *To construct an equilateral triangle on a given right line* A B.

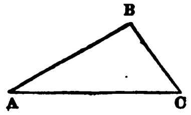


Bisect A B in C, at which erect the perpendicular C D. Take C O equal to A C; join A O, and in C B produced take C E equal to A O. Draw O E to which make C D equal, and join A D, B D; the triangle A D B is that required.

For $AO^2 = 2AC^2$ and $EO^2 = 3AC^2$ (54); wherefore $AD^2 = 4AC^2$, and $AD = 2AC$ or A B. The sides A B, A D, B D being therefore equal, the triangle A B D is isosceles.

PROBLEM VIII.

110. *To construct a triangle of which two sides* A B, B C *and the included angle* B *are given.*

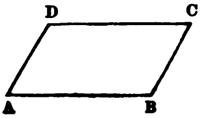


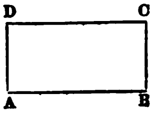
The construction of this is immediately effected by problem II. or V., according as the given angle B is a right or an oblique angle.

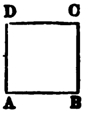
Remark. Should the angles A, B and the adjacent side A B be given, the corresponding triangle may also be constructed by problem V.

PROBLEM IX.

111. *To construct a parallelogram of which two sides* A B, B C *and the included angle* B *are given*







Make A B C equal to the given angle, and draw A D parallel and equal to B C. Join C D, and A B C D will evidently be the required parallelogram.

Remark. When B is a right angle, the corresponding rectangle may be constructed by the perpendicular B C and parallel A D; and in like manner may a square be constructed on a given line A B.

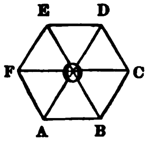
PROBLEM X.

112. *To construct a square equal to two given squares.*

A right angled triangle, of which the base and perpendicular are the given sides of the squares, may be constructed by prob. VIII. The hypotenuse of this triangle will be the side of the required square (87), which may be constructed in the manner noticed in the preceding remark.

PROBLEM XI.

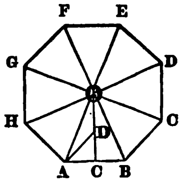
113. *To construct a regular hexagon on a given right line* A B.



Construct the equilateral triangle A O B on the given line A B. This triangle applied successively round O will evidently form the required hexagon A B C D E F.

PROBLEM XII.

114. *To construct a regular octagon on a given right line* A B.



Bisect A B in C; erect the perpendicular C D equal to

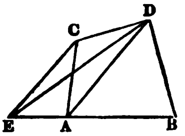
A C; join A D which apply in C D produced, from D to O. Draw A O, B O, forming with A B the isosceles triangle A O B, which applied successively round O will form the required octagon.

The angle A D C is evidently half a right angle; and the triangle A D O being isosceles, this angle is equal to twice the interior opposite angle A O D (cor. 31); and therefore equal to A O B, eight of which constitute four right angles about the central point O. The figure A C D E H is therefore a regular octagon.

The preceding are the most useful regular polygons. The construction of the pentagon and decagon will be given in the second book.

PROBLEM XIII.

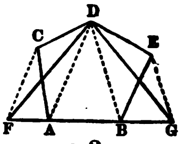
115. *To transform a given quadrilateral* A B C D *into a triangle having an equal surface.*



Draw the diagonal A D, parallel to which through C draw the right line C E, meeting B A produced in E. Join D E; and B D E is the required triangle.

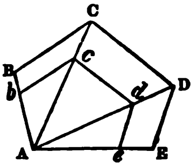
For the triangles A C D, A E D being on the same base A D and between the same parallels A D, E C, are equal (cor. 63); wherefore the triangle B D E is equal to the quadrilateral A B C D.

Remark. In a similar manner, by drawing diagonals and parallels, any rectilinear figure may be converted into an equal triangle.



PROBLEM XIV.

116. *To make a rectilinear figure similar to a given rectilinear figure* A B C D E.



Draw diagonals from any angular point, as A; and parallel to the sides B C, C D, D E draw *b c*, *c d*, *d e*, forming a figure which is evidently similar to the given figure A B C D E.

Remark. By the preceding construction, A B C D E may be said to be proportionally contracted; it may be enlarged by producing the sides A B, A E and the diagonals A C, A D.

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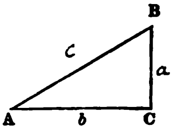
§ 2. OF THE MENSURATION OF RECTILINEAR FIGURES.

PROBLEM I.

117. *To determine the altitude of a right angled triangle, of which the base and hypotenuse are given.*



The altitude $a=\sqrt{\{(c+b)\ (c-b)\}}$ ..... [1].

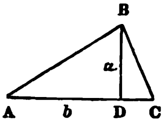
For $BC^2=AB^2-AC^2$ (cor. 54), or $a^2=c^2-b^2=(c+b)$ $(c-b)$; wherefore $a=\sqrt{\{(c+b)\ (c-b)\}}$ (th. 3).

Example. Let $b=20$ feet, $c=36$; then $a=\sqrt{(56.16)}=$ $8\sqrt{14}=29,9332$.

Remark. By interchanging a and b, the base may be found in a similar manner, from the altitude and hypotenuse.

PROBLEM II.

118. *To find the area of a triangle, of which the base and altitude are given.*



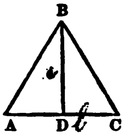
The area=$\frac{1}{2}\,ab$, ... [2],
in which a denotes the altitude B D, and b the base A C.

Example. Let a=4 feet 6 inches, and b=10 ft. 3 in.

The area=$\frac{1}{2}.\overset{ft.}{4},\ \overset{in.}{6}\times\overset{ft.}{10},\ \overset{in.}{3}$=23 ft. 0 in. 9 parts, by cross multiplication.

PROBLEM III.

19. *To determine the altitude of an equilateral triangle of which the side is given.*



The altitude $a=\frac{1}{2}\sqrt{3}.b$, ... [3],
in which b denotes the base.

For A D=$\frac{b}{2}$, and therefore $a=\sqrt{\{\frac{3\,b}{2}.\frac{b}{2}\}}=\frac{1}{2}\sqrt{3}.b$.

Example. Let b=12 feet; then $\sqrt{3}$=1, 73205 and a= 10, 3923 feet.

PROBLEM IV.

120. *To find the area of an equilateral triangle.*

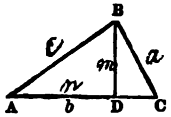
The area=$\frac{1}{4}\sqrt{3}.b^2$, ... [4],
in which b denotes the base.

For the area=$\frac{1}{2}\,ab$, and B D=$\frac{1}{2}\sqrt{3}.b$; therefore the area =$\frac{1}{4}\sqrt{3}.b^2$.

Example. Let b=12 feet. Then the area=62.3538 square feet=62 feet 4 in. $\frac{1}{4}$ nearly.

PROBLEM V.

121. *To determine the altitude of a triangle of which the three sides are given.*



The altitude $m=\frac{2}{b}\sqrt{\{p\,(p-a)\,(p-b)\,(p-c)\}}$, . . . [5], in which p denotes the semiperimeter $\frac{1}{2}\,(a+b+c)$; and a, b, c the sides opposite to the angular points A, B, C.

For if the segment A D$=n$, $m=\sqrt{\{(c+n).(c-n)\}}$; and since $a^2=b^2+c^2-2\,b\,n$ (55), there results $n=\frac{b^2+c^2-a^2}{2\,b}$, and $c+n=\frac{(b+c)^2-a^2}{2\,b}=\frac{(a+b+c)\,(b+c-a)}{2\,b}$, $c-n=\frac{a^2-(b-c)^2}{2\,b}$ $=\frac{(a+c-b)(a+b-c)}{2\,b}$. But $a+b+c=2p$, $b+c-a=2(p-a)$, &c. Wherefore $c+n=\frac{2}{b}.p(p-a)$; $c-n=\frac{2}{b}.(p-b)\,(p-c)$; and by substitution there will result formula 5.

PROBLEM VI.

122. *To find the area of a triangle of which the sides are given.*

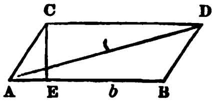
The area$=\sqrt{\{p\,(p-a)\,(p-b)\,(p-c)\}}$, ... [6], in which p denotes the semiperimeter and a, b, c the sides.

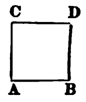
For if m be the altitude, the area$=\frac{1}{2}\,b\,m$; and $m=\frac{2}{b}.$ $\sqrt{\{p\,(p-a)\,(p-b)\,(p-c)\}}$, by the last problem; wherefore this area is expressed by formula 6.

Example. Let $a=6$, $b=12$, and $c=8$ yards; then $p=13$, $p-a=7$, $p-b=1$, $p-c=5$; and the required area$=\sqrt{(13.7.1.5)}=\sqrt{455}=21,31$ square yards.

PROBLEM VII.

123. *To find the area of a parallelogram, of which the base and altitude are given.*





The area$=ab$ . . . . [7],

in which a is the altitude and b the base.

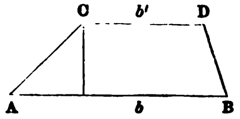
For the area is equal to that of a rectangle of the same altitude a and base b; and is therefore expressed by formula [7].

Example. Let $a=8$ feet 3 inches, and $b=12$ ft. 5 in. The area$=\overset{ft.}{8}, \overset{in.}{3}\times\overset{ft.}{12}, \overset{in.}{5}=102$ feet, 5 inches, 3 parts, square measure.

Remark. When the base and altitude are equal, the figure becomes a square A B C D, of which the area$=b^2$.

PROBLEM VIII.

124. *To find the area of a trapezoid, of which the altitude and two parallel sides are given.*



The area$=\frac{1}{2}a(b+b')$ . . . . [8],

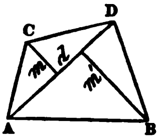
in which a is the altitude and b, b' the opposite parallel sides.

For the area is half that of a parallelogram of the same altitude a, and of which the base is equal to the sum of the parallel sides b, b'; and is therefore expressed by formula [8].

Example. Let $a=6$ feet 2 inches, and $b=\overset{ft.}{12}, \overset{in.}{3}$; $b'=\overset{ft.}{8}, \overset{in.}{4}$. The area $=\overset{ft.}{3}, \overset{in.}{1}\times\overset{ft.}{20}, \overset{in.}{7}=\overset{ft.}{63}, \overset{in.}{5}, \overset{p.}{7}$.

PROBLEM IX.

125. *To find the area of a quadrilateral, of which a diagonal and perpendiculars to it from the opposite angles are given.*



The area$=\frac{1}{2}d\,(m+m')$ .... [9],

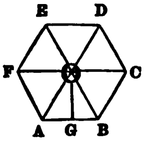
in which d is the given diagonal, and m, m' the corresponding perpendiculars from the opposite angles.

For the areas of the component triangles A C D, A B D are $\frac{1}{2}\,d\,m$ and $\frac{1}{2}\,d\,m'$; and therefore the area of the quadrilateral is expressed by formula [9].

Example. Let $d=10$ yards, $m=3$, and $m'=6$. Then the area$=\frac{1}{2}.10.9=45$ square yards.

PROBLEM X.

126. *To find the area of a regular polygon, of which a side and the corresponding perpendicular from the central point are given.*



The area$=m\,p$ .... [10],

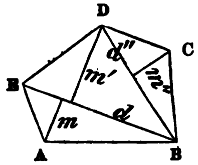
in which p is the semiperimeter and m the perpendicular O G

Example. Let A B$=20$ feet be the side of a regular hexagon, in which case $p=\overset{\text{yds.}}{60}$, $m=\sqrt{(\text{A O}^2-\text{O G}^2)}=\sqrt{300}=$ 17,3205, and the area$=$1039,238 square yards.

Remark. Had the polygon been an octagon, we should have had $p=80$, $m=10+\sqrt{200}=24{,}1421$; and the area$=$ 1931,368.

PROBLEM XI.

127. *To find the area of an irregular polygon, determined by diagonals and perpendiculars.*



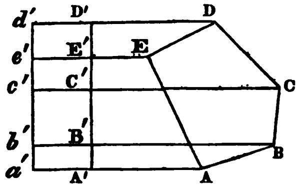
$$\text{The area} = \tfrac{1}{2}\{d(m+m')+d''m''\} \ldots\ldots [11].$$

The figure A B C D E being composed of a quadrilateral and triangle, its area is expressed by formula [11].

Example. Let $d=10$, $m=3$, $m'=6$; and $d''=9$, $m''=4$. The area$=\frac{1}{2}(10.9+9.4)=63$.

PROBLEM XII.

128. *To find the area of any rectilinear figure, having given the several perpendiculars drawn from its angular points to a right line fixed in position, and also the parts of this line intercepted by those perpendiculars.*



$$\text{The area} = \tfrac{1}{2}\{(mn+m'n'+m''n'')-(m'''n'''+m^{iv}n^{iv})\} \ldots [12],$$

in which m, m', &c. denote A′ B′, B′ C′, &c. intercepted on A′ D′, by the perpendiculars A A′, B B′, &c., and n, n', &c., represent the sums of every two successive perpendiculars; as A A′+B B′, B B′+C C′, &c.

The figure A B C D E is the difference of A′A B C D D′ and D′D E A A′; the first of which is composed of the three trapezoids A A′B B′, B B′C C′, C C′D D′; the second of the two trapezoids D D′E E′, E E′A A′. The area of A A′B B′$=\frac{1}{2}$A′ B′.(A A′+B B′); the area of B B′C C′=

$\frac{1}{2}$ B′ C′.(B B′+C C′); &c. Wherefore the area of A B C D E is equal to the expression [12].

Remark. The intercepted parts m''', m^{iv} of A′ D′, corresponding to the subtractive trapezoids, are in directions opposite to m, m', m'', corresponding to the remaining trapezoids, so that $(m+m'+m'')-(m'''+m^{iv})=0$; and this observation will suffice to change the form of [12] and render it applicable to any rectilinear figure whatever.

It should also be observed, that if a right line $a'\,d'$ be drawn parallel to A′ D′; and the figure A B C D E be referred by perpendiculars to this line instead of A′ D′, the difference between the expression for its area and [12] will be $\frac{1}{2}\{(m\,\mathrm{N}+m'\,\mathrm{N}+m''\,\mathrm{N})-(m'''\,\mathrm{N}+m^{iv}\,\mathrm{N})\}$, in which $\mathrm{N}=a'\,\mathrm{A}'$; and which is nothing by virtue of $(m+m'+m'')-(m'''+m^{iv})=0$, above given. The different factors n, n', &c. in [12] may therefore be increased or diminished by any number N taken at pleasure, without affecting the resulting difference corresponding to A B C D E.

Example.

yds.	yds.			
Let A A′=14	A′ B′= 4	$n=38$	$m\,n=152$	
B B′=24	B′ C′= 7	$n'=50$	$m'\,n'=350$	
C C′=26	C′ D′= 8	$n''=44$	$m''\,n''=352$	854
D D′=18	D′ E′= 4	$n'''=25$	$m'''n'''=100$	
E E′= 7	E′ F′=15	$n^{iv}=21$	$m^{iv}\,n^{iv}=315$	415
				439

The area of A B C D E=219$\frac{1}{2}$ square yards.

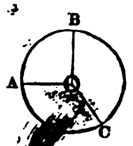
If n, n', &c. be increased by 8; the area of A B C D E= $\frac{1}{2}\{(184+406+416)-(132+435)\}=219\frac{1}{2}$; if n, n', &c. be diminished by 21, the area$=\frac{1}{2}\{68+203+184-16\}=219\frac{1}{2}$; and if n, n', &c. be algebraically diminished by 44, so that the first, fourth, and fifth remainders may be negative, the area$=\frac{1}{2}\{-24+42+76+345\}=219\frac{1}{2}$, as before.

BOOK II.

OF THE CIRCLE.

1. **When** a right line revolves in a plane about one of its extremities considered as fixed, the other extremity, during its entire revolution, describes a curve line enclosing a plane surface, and forms a curvilinear figure, which is characterized by its perfect regularity and the uniformity of its boundary.

2. A *circle* is a plane figure contained by a curve line, such that all right lines drawn to it from an interior point, are respectively equal.



The figure A B C is a circle; the bounding curve A B C A is called the *circumference;* the interior point O is named the *centre*, and any of the equal right lines O A, O B, O C is termed a *radius.*

3. The position of a circle is evidently determined by the position of its centre; and its magnitude depends on its radius, which may be equal to any given right line, or may extend to any required distance from the centre.

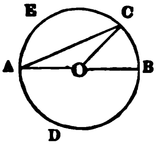
CHAPTER I.

OF RIGHT LINES DRAWN IN AND ABOUT A CIRCLE.

4. **The** circumference of a circle being concave towards any interior point, a right line will evidently meet or cut it in two points, and will divide it and the circle into parts which may be respectively equal or unequal.

5. A *diameter* is a right line which passes through the centre and terminates in the circumference, as A B: a

diameter is twice a radius; it divides the circle into parts which become identical when applied to each other, and which are therefore *semicircles* having *semicircumferences* for their curve boundaries; as A B C A, or A B D A.



6. A *chord* is a right line which terminates in the circumference, but does not pass through the centre of a circle; as A C. Each of the unequal parts formed in the circle by a chord, is called a *segment;* the curve boundary of which is named an *arc;* as A C E A or C B D A C. A chord is evidently less than two radii at its extremities. It is therefore less than a diameter, and may be diminished at pleasure by increasing its perpendicular distance from the centre of the circle. A chord is said to subtend the arc which it terminates; and this arc, and the radii at its extremities, enclose a part of the circle, and constitute a trilateral figure which is called a *sector;* as B O C B.

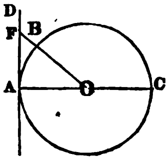
7. It is evident from the uniformity of the circle, that equal arcs terminate equal chords, and constitute the curve boundaries of equal segments and sectors.

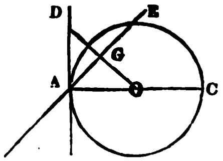
---

§ 1. OF THE POSITION OF RIGHT LINES WHICH MEET A CIRCLE.

THEOREM I.

8. *The perpendicular drawn to the diameter of a circle from either of its extremities, falls without the circle; and any right line diverging from that perpendicular at the extremity of the diameter, cuts the circle.*





Let A B C be a circle, A C a diameter, A D a perpendicular to A C, and A E a right line diverging from A D at the point A; A D falls without the circle A B C, and A E cuts it.

In A D, take any point F; join O F, and let this right line meet the circle in B. Because O A F is a right angled triangle, the hypotenuse O F is greater than O A, which is equal to the radius O B; wherefore the point F is without the circle A B C. But F is any point in A D; wherefore A D falls without the circle A B C.

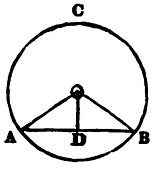
Again, let O G be perpendicular to A E. In the right angled triangle O G A, the side O G is less than the hypotenuse or radius O A; the point G is therefore within the circumference, and A E cuts the circle A B C.

9. A *tangent* to a circle is a right line which meets, but does not cut the circle. The point in which the tangent meets the circle, is called the *point of contact.*

Corollary. A perpendicular to the diameter of a circle at its extremity, is a tangent to the circle at this point.

THEOREM II.

10. *The perpendicular drawn from the centre of a circle to a chord, bisects this chord.*



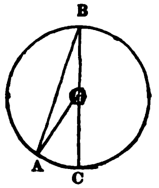
Let A B C be a circle, A B a chord, and O D a perpendicular to this line; A D is equal to B D.

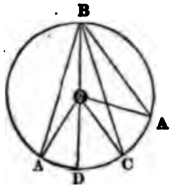
Draw the radii O A, O B. Because the triangle A O B is isosceles, the angle A is equal to B; and O D being perpendicular to A B, the right angled triangles A O D, B O D are equal in all respects; and therefore A D is equal to B D.

Remark. The converse of this theorem is also true, and may be thus expressed: *A right line which bisects a chord perpendicularly passes through the centre of the circle.*

THEOREM III.

11. *The angle at the centre of a circle is twice the angle at the circumference, when those angles are subtended by the same arc or chord.*

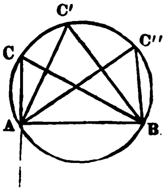




Let A O C, A B C be angles at the centre and circumference of the circle A B C A; the angle A O C is twice A B C.

The centre O may be in one of the chords A B, B C, or within or without these chords. In the first case, the angle A O C is equal to the equal angles A, B of the isosceles triangle A O B (cor. 31); and is therefore twice the angle B or A B C. In the second and third cases, let B O D be a diameter. The angles A O D, D O C are, by the first case, twice the angles A B D, D B C; wherefore the whole or the remaining angle A O C is equal to the whole or the remaining angle A B C.

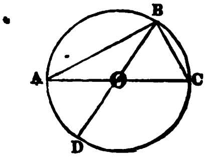
Corollary. All the angles which are in the same segment, or in equal segments of a circle, are equal; for each of them is equal to one and the same angle at the centre.



Remark. By virtue of this corollary, the arc of any segment of a circle, capable of containing a given angle, may be described *mechanically*, when the extremities of its chord are given. For the sides of the *invariable* angle A C B being moved through the fixed points A, B, will take different positions as A C′, C′B; A C″, C″B; &c., corresponding to which the vertices C′, C″, &c. will be in the arc A C′B.

THEOREM IV.

12. *The angle in a semicircle is a right angle.*



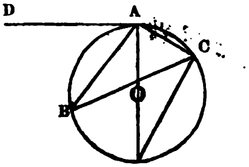
Let A B C A be a semicircle, in which the angle A B C is formed by chords at the extremities of the diameter; A B C is a right angle.

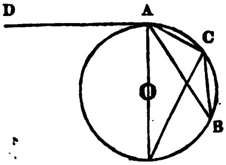
Through B draw the diameter B O D. The angles A O D, D O C are respectively equal to twice the angles A B D, D B C (11). But the angles A O D, D O C are together equal to two right angles (14, I); wherefore the angles A B D, D B C are together equal to one right angle; and A B C is therefore a right angle.

Remark. By means of this theorem, a perpendicular may be drawn to a right line from a given point.

THEORE[illegible].

13. *The angle formed by a chord and tangent at any point of the circumference of a circle, is equal to the angle in the alternate segment, corresponding to that chord.*





Let A B C be a [illegible]le of which A B is a chord, and A D a tangent at A; the angle B A D is equal to A C B.

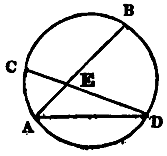
Draw the diameter A E, and join C E. Because A D is a tangent and A E a diameter, the angle D A E is a right angle (cor. 9), and also the angle A C E (12). But the angle B A E is equal to B C E, subtended by the same arc B E (11); wherefore the remaining or the whole angle B A D is equal to A C B; and since A C B is equal to any angle in the segment A B C A, the angle B A D is equal to the angle in this segment.

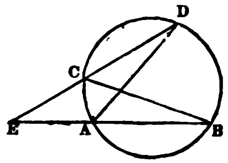
Remark. By virtue of this and the preceding theorem, a segment of a circle capable of containing a given angle, may be described on a given chord. For the position of the tangent relative to the chord becomes known from the given angle; and thence the position and length of the diameter may be determined.

14. A *secant* is a right line drawn from an exterior point to cut a circle and terminate in its circumference.

THEOREM VI.

15. *The angle formed by two chords or two secants which intersect each other within or without a circle, is equal to the sum or difference of the angles at the circumference, and which are subtended by the intercepted arcs.*





Let A B, C D be two chords or two secants, intersecting in E within or without the circle A B D; the angle B E D is equal to the sum or difference of the angles B A D, A D C, subtended by the arcs B D, A C.

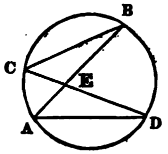
Join A D. In case of the chords, the angle B E D is equal to the interior and opposite angles A, D of the triangle A D E; and which are subtended by the arcs B D, A C.

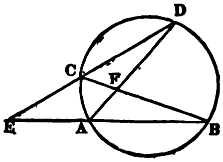
In case of the secants, the angle B A D is equal to the angles D, E of the triangle A D E; wherefore the angle B E D is the difference of the angles B A D and D, which are subtended by the arcs B D, A C.

§ 2. OF THE PROPORTION OF RIGHT LINES MEETING A CIRCLE.

THEOREM VII.

16. *If two chords or two secants intersect each other within or without a circle, the product of their segments between their point of intersection and the circumference, are respectively equal.*





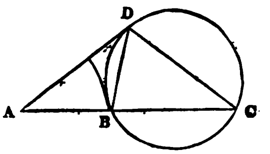
Let A B, C D be two chords or two secants, intersecting each other in E within or without the circle A B D; the product $EA.EB=EC.ED$.

Join A D, B C. The triangles A E D, B E C having the angles B, D equal (cor. 11), and also the angles A E D, B E C, are equiangular; and $EA : EC :: ED : EB$, whence $EA.EB=EC.ED$.

Remark. The diameter may evidently take place of either chord as A B, in which case, if it be perpendicular to the intersected chord C D, the product of its segments is equal to the square of half this chord, as $AE.EB=CE^2$.

THEOREM VIII.

17. *If a secant and tangent to a circle meet each other, the product of the secant and its exterior segment is equal to the square of the tangent.*



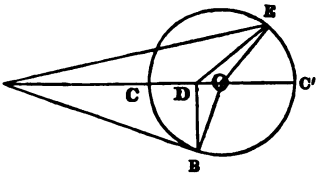
Let A B C and A D be a secant and tangent to the circle B C D; $AC.AB=AD^2$.

Join B D, C D. The triangles A B D, A D C having the angle A common, and the angles A D B, A C D equal (13), are equiangular; and A B : A D :: A D : A C; whence A C.A B=A D^2.

Remark. By means of this theorem, A D may be divided in extreme and mean ratio at E, so that A E^2=A D.D E. For if A E, A D be respectively equal to A B, A C, the expression A C.A B=A D^2 becomes A E.(A E+A D)= A D.(A E+D E), from which there results A E^2=A D.D E. It is evident that both A D and B C may be taken equal to the diameter of the circle.

THEOREM IX.

18. *If a tangent to a circle meet a secant passing through the centre, and a perpendicular be drawn to the secant from the point of contact, the tangent and perpendicular are proportional to two right lines drawn from their extremities in the secant, to any point in the circumference.*



Let A B, A C be a tangent and secant to the circle of which the centre is O, and let B D be a perpendicular to A C, and A E, D E two right lines meeting in the circumference; A B : D B :: A E : D E.

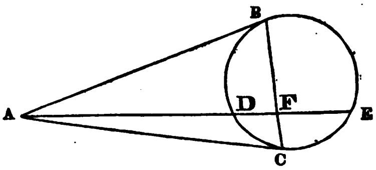
Draw the radius O E. The right angled similar triangles A B O, B D O give A B : B D :: A O : B O :: B O : D O; from which A O : E O :: E O : D O, since E O=B O. The triangles A O E, E O D are therefore similar (51, I); and give A E : D E :: E O : D O. But it has been proved that A B : B D :: B O or E O : D O; wherefore A B : B D :: A E : D E.

Corollary. Hence A B : B D :: A C : D C :: A E : D E, so that right lines joining B C, E C will bisect the angles A B D, A E D (57, I). From the proportion A O : B O ::

B O : D O, first proved, there results $AO.DO=BO^2=CO^2$, which will determine the point A when D is given, and the contrary.

THEOREM X.

19. *If two tangents and a secant diverge from a point without a circle, and a chord join the points of contact, the product of the secant and its middle part is equal to the product of its extreme parts, terminating in the exterior point and in the circumference.*



Let A B, A C, and A D E be two tangents and a secant to the circle B C D; and B C a chord joining the points of contact, and intersecting A E in F; the product $AE.DF=AD.EF$.

The tangents A B, A C are evidently equal, and the triangle A B C is isosceles; wherefore $AB^2=AF^2+BF.CF$ (cor. 58, I). But $AB^2=AD.AE$ (17), and $BF.CF=DF.EF$ (16); wherefore $AD.AE=AF^2+DF.EF$. Now $AD.AE=AD.AF+AD.EF$; and $AF^2=AF.(AD+DF)=AF.AD+AF.DF$; wherefore $AE.DF=AD.EF$.

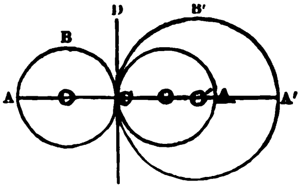
Corollary. The secant A E is divided harmonically in the points D, F.

---

§ 3. OF TWO TANGENTIAL CIRCLES.

20. When two circles have a common tangent, and the same point of contact, their circumferences meet at this point, without cutting each other; and the distance between their centres is evidently either the sum or the difference of their

radii, according as these centres are on different sides, or on the same side of the point of contact.



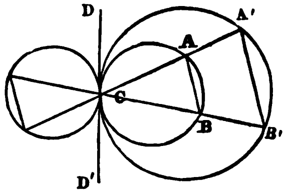
The circles A B C, A′ B′ C are in contact at the point C. The distance O O′=O′ C±O C, according as the centres O, O′ are on different sides, or on the same side of C; in which situations, the circles obviously touch each other either exteriorly or interiorly.

It appears hardly necessary to observe that when O O′ is greater than the sum O′ C+O C; or less than the difference O′ C—O C, the circumferences of the circles will be separated, and not in contact; and that when O O′ is less than O′ C+O C or greater than O′ C—O C, the circles will intersect and have a common part bounded by arcs of their circumferences.

§ 4. OF THE POSITION OF CHORDS IN TANGENTIAL CIRCLES.

THEOREM XI.

21. *If two chords diverge from the point at which two circles are in contact, they intercept arcs of which the corresponding chords are respectively parallel.*



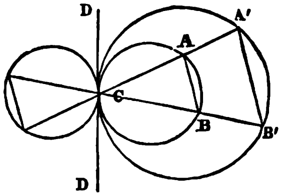
Let A B C, A′ B′ C be two circles in contact at C; and let C A A′, C B B′ be chords intercepting the arcs A B, A′ B′; the chords A B, A′ B′ joining these arcs are parallel.

At the point of contact C, let D C D′ be a tangent common to both circles A B C, A′ B′ C. The angle A C D formed by the chord A C and tangent C D, is equal to the angle A B C in the alternate segment (13); and the same angle A C D, or the angle A′ C D is equal to A′ B′ C. Wherefore the angle A B C is equal to A′ B′ C; and these being exterior and interior angles relatively to C B′, the chords A B, A′ B′ are parallel (cor. 23, I).

§ 5. OF THE PROPORTION OF RIGHT LINES IN TANGENTIAL CIRCLES.

THEOREM XII.

22. *Two chords diverging from the point in which two circles touch each other, are cut proportionally by the circumferences.*



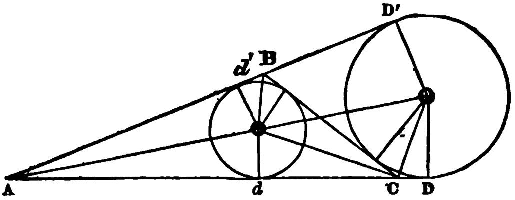
For chords joining the intercepted arcs are parallel (21); and these and the diverging chords form similar triangles, the sides of which or the diverging chords are respectively proportional.

THEOREM XIII.

23. *If two circles touch three right lines forming a triangle, the radii of the greater and less circle are proportional to the semiperimeter of the triangle, and its excess above that side which separates the circles; and the product of these radii is equal to the product of the excesses of the semiperimeter above the two other sides of the triangle.*

Let O D D′, *o d d′* be two circles in contact with the three right lines A B, B C, C A, forming the triangle A B C; and let O D, *o d* be radii perpendicular to A D the common tangent;

$\mathrm{OD} : od :: p : p - \mathrm{BC}$, and $\mathrm{OD}.od = (p - \mathrm{AC})\,(p - \mathrm{AB})$, in which p represents the semiperimeter $\frac{1}{2}\,(\mathrm{AB} + \mathrm{BC} + \mathrm{CA})$.



From the centres O, o to the several points of contact, let radii be drawn, and join O C, o C. The tangents at each of the vertices A, B, C being respectively equal, it follows that $\mathrm{AD} = p$, $\mathrm{A}d = p - \mathrm{BC}$, $\mathrm{CD} = p - \mathrm{AC}$, $\mathrm{C}d = p - \mathrm{AB}$; and the triangles A O D, A $o\,d$ being evidently similar $\mathrm{OD} : od :: \mathrm{AD} : \mathrm{A}d$, or $\mathrm{OD} : od :: p : p - \mathrm{BC}$. The angles B C D, B C d are together equal to two right angles; and they are bisected by O C, o C; wherefore the angles O C D, o C d are equal to a right angle; and the two right angled triangles O C D, o C d being similar give $\mathrm{OD} : \mathrm{C}d :: \mathrm{CD} : od$, or $\mathrm{OD} : (p - \mathrm{AB}) :: (p - \mathrm{AC}) : od$, from which there results $\mathrm{OD}.od = (p - \mathrm{AC})\,(p - \mathrm{AB})$.

Corollary. The first proportion gives $\mathrm{OD}.(p - \mathrm{BC}) = od.p$; which combined with the preceding equation, there will result the two formulæ:

$$\mathrm{OD}^2.(p - \mathrm{BC}) = p\,(p - \mathrm{AC})\,(p - \mathrm{AB}),$$

$$od^2.p = (p - \mathrm{BC})\,(p - \mathrm{AC})\,(p - \mathrm{AB}),$$

by virtue of which the radii O D, $o\,d$ may be determined when the sides of the triangle A B C are given.

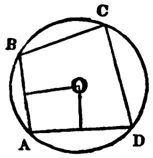
Remark. The area of the triangle A B C in terms of its sides may be found from the second of these formulæ. This area is equal to the areas of the three component triangles A o B, B o C, C o A; and therefore equal to $od.p$, which is $\sqrt{\{p(p - \mathrm{BC})\,(p - \mathrm{AC})\,(p - \mathrm{AB})\}}$, according to the formula just mentioned. See 122, I.

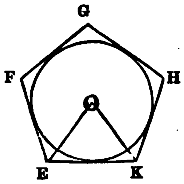
CHAPTER II.

OF RECTILINEAR FIGURES DESCRIBED IN AND ABOUT CIRCLES.

24. When three or more successive chords are drawn in a circle, they form a rectilinear figure, which has its angular points in the circumference; and when three or more successive tangents are drawn to a circle, they form a rectilinear figure, each side of which meets without cutting the circumference.

25. A rectilinear figure *inscribed in a circle* is that of which the angular points are in the circumference. A rectilinear figure *described about a circle* is that of which each side is a tangent to the circumference.





The quadrilateral A B C D is inscribed in the circle of which O is the centre. Its sides are bisected by perpendiculars drawn to them from this point (10); and any two bisecting perpendiculars drawn to adjacent sides, evidently determine the centre of the circumscribing circle.

Corollary. Any triangle may be circumscribed by a circle.

The pentagon E F G H K is described about the circle of which the centre is O. Its angles are bisected by right lines drawn from this point; and any two bisecting lines drawn from consecutive vertices, evidently determine the centre of the inscribed circle.

Corollary. Any triangle may have an inscribed circle.

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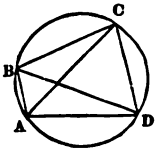
§ 1. OF THE ANGLES AND SIDES OF INSCRIBED FIGURES.

THEOREM I.

26. *The opposite angles of a quadrilateral inscribed in a circle, are together equal to two right angles.*

Let A B C D be a quadrilateral inscribed in the circle A C D; the opposite angles A B C, A D C are together equal to two right angles.



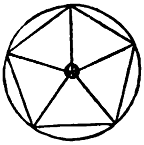
Draw the diagonals A C, B D. In the triangle A B C, the angles B A C, A C B are respectively equal to B D C, A D B, which are together equal to the angle A D C; wherefore the three angles A B C, B A C, A C B are equal to the angles A B C, A D C. But these three angles are together equal to two right angles (31, I); wherefore the angles A B C, A D C are together equal to two right angles.

Corollary. The exterior angle formed by producing any side of an inscribed quadrilateral, is equal to the interior opposite angle. For these angles are supplements of the same interior angle, and are therefore equal to each other.

Remark. When either of the opposite angles, as A B C, is increased or diminished by diminishing or increasing either or both of its sides, the vertex B will fall within or without the circumference, which passes through the remaining vertices; and the quadrilateral thus having its opposite angles either greater or less than two right angles, cannot be inscribed in a circle.

THEOREM II.

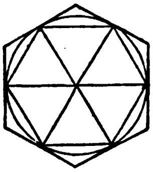
27. *An equilateral polygon inscribed in a circle, is a regular polygon.*

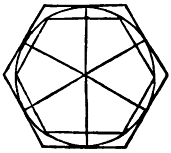


For the equal sides of the polygon become the bases of equal isosceles triangles, formed by radii at their extremities; and the several angles at these bases being equal, the angles of the polygon are therefore equal, and the polygon is regular.

THEOREM III.

28. *A regular polygon being inscribed in a circle, a similar polygon may be described about the circle; and these polygons may have their sides respectively parallel.*





First, let radii be drawn to the angular points of the inscribed polygon, and let tangents to the circumference pass through their extremities. These radii and tangents being perpendicular to each other (cor. 9), form quadrilaterals which are equiangular, and evidently equal in all respects; and which therefore constitute a regular polygon about the circle. But this and the inscribed polygon have the same number of sides; they are therefore similar polygons.

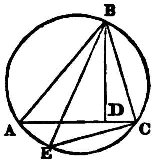
Secondly, let radii bisect the sides of the inscribed polygon, and let tangents to the circle pass through their extremities. These radii being perpendicular to the corresponding sides and tangents, form in connexion with these lines the same number of interior and exterior equiangular quadrilaterals. But the interior quadrilaterals composing the inscribed polygon, are evidently equal in all respects. Wherefore the exterior quadrilaterals are also equal, and compose a regular polygon, the sides of which and those of the similar inscribed polygon are respectively parallel.

Remark. The converse of this theorem is also true, as to the possibility of inscribing a regular polygon similar to one described about a circle.

§ 2. OF THE PROPORTION OF THE SIDES OF INSCRIBED FIGURES.

THEOREM IV.

29. *The product of any two sides of a triangle inscribed in a circle, is equal to the product of the perpendicular to the third side, and the diameter of the circle.*

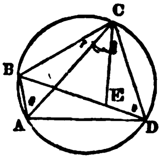


Let A B C be a triangle in which B D is the perpendicular to the base A C, and let B E be the diameter of the circle; A B.B C=B D.B E.

Join C E. The right angled triangles A B D, E B C (12), having the angles B A C, B E C equal (cor. 11), are similar, and A B : B E :: B D : B C; whence A B.B C=B D.B E.

THEOREM V.

30. *The product of the diagonals of a quadrilateral inscribed in a circle, is equal to both the products of its opposite sides.*



Let A B C D be a quadrilateral inscribed in the circle A C D, and of which A C, B D are the diagonals; A C.B D= A B.C D+B C.A D.

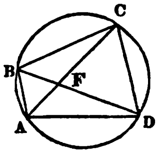
At the point C, let the angle D C E be equal to A C B; and therefore the greater angle A C D equal to B C E. The triangles A C B, D C E having the angles A C B, D C E equal, and also the angles C A B, C D E (cor. 11), are similar, and A C : C D :: A B : D E; whence A C.D E=A B.C D.

The triangles A C D, B C E having the angles A C D, B C E equal, and also the angles C A D, C B E (cor. 11), are similar, and A C : B C :: A D : B E; whence A C.B E=B C.A D. Add this to A C.D E=A B.C D already obtained, and there results A C.B D=A B.C D+B C.A D.

Corollary. If A D be equal to C D, the triangle A C D will be *isosceles*, and there will result A C.B D=C D.(A B+B C); and if A C be also equal to C D, the triangle A C D will be *equilateral*, and the preceding result will become simply B D=A B+B C.

THEOREM VI.

31. *The diagonals of a quadrilateral inscribed in a circle, are proportional to the sums of the products of the sides which diverge from their extremities.*



Let A B C D be a quadrilateral inscribed in the circle A C D, and of which A C, B D are the diagonals; A C : B D :: B A.A D+B C.C D : A B.B C+A D.D C.

The triangles B F A, C F D are evidently similar, and also the triangles A F D, C F B; wherefore B F : C F :: B A : C D; A F : B F :: A D : B C; and C F : D F :: B C : A D. The products of the first of these proportions and each of the others give A F : C F :: B A.A D : B C.C D, and B F : D F :: A B.B C : A D.D C, which are remarkable relations, and from which there results by composition of ratios, A C : C F :: B A.A D+B C.C D : B C.C D and B D : D F :: A B.B C+A D.D C : A D.D C. But the second and fourth terms of these constitute the proportion C F : D F :: B C : A D already obtained; wherefore the remaining terms are proportional, and A C : B D :: B A.A D+B C.C D : A B.B C+A D.D C.

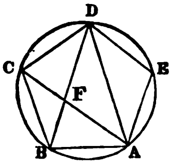
Corollary. If this and the preceding theorem be briefly written A C : B D :: L : M, and A C.B D=N, there will result the two formulæ

$$AC^2.M=L.N \text{ and } BD^2.L=M.N,$$

from which the diagonals may be determined, when the sides of the quadrilateral are given.

THEOREM VII.

32. *The square of the side of a regular pentagon inscribed in a circle, is equal to the product of the chord subtending any of the angles, and the excess of that chord above the side of the pentagon.*



Let A B C D E be a regular pentagon inscribed in the circle A C E, and let B D be a chord subtending the angle B C D; $AB^2=BD.(BD-AB)$.

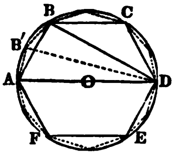
Join A C, A D, and let A C meet B D in F. The triangles A B D, B F A having the angle A B D common, and the angles A D B, B A C equal (cor. 11), are equiangular; and A B : B D :: B F : A B; whence $AB^2=BD.BF$. But A B F being similar to A B D, is an isosceles triangle, and also A F D; wherefore D F=A B, B F=B D−A B, and $AB^2=BD.(BD-AB)$.

Corollary. The chord B D is divided in extreme and mean ratio at F (cor. 57, I); and the side A B of the pentagon is equal to the greater segment D F.

Remark. If radii be drawn parallel to A D, B D, the chord joining their extremities will evidently be the side of a regular decagon; and will be the greater segment of the radius divided in extreme and mean ratio by a right line parallel to A F.

THEOREM VIII.

33. *If a regular polygon be inscribed in a circle, and chords be drawn from the extremities of two adjacent sides to meet in the circumference, the square of the side of the polygon, together with the product of the two extreme chords, is equal to the square of the intermediate chord.*



Let AB, BC be two sides of a regular polygon inscribed in the circle ACE; and let AD, BD, CD be chords meeting in the circumference; $AB^2+AD.DC=BD^2$.

In the quadrilateral $ABCD$, of which BD is a diagonal; $BD^2.L=M.N$ (cor. 31); and since $AB=BC$, there results in the present case $L=N$ and $M=AB^2+AD.DC$; wherefore $AB^2+AD.DC=BD^2$.

Corollary. When AD is a diameter, ABD is a right angled triangle, and $AD^2=AB^2+BD^2$, which combined with $AB^2+AD.DC=BD^2$; there results the expressions $2AB^2=AD.(AD-DC)$, and $2BD^2=AD.(AD+DC)$.

Remark. These formulæ are important as respects inscribed regular polygons. By means of them and the known side of any regular inscribed figure, the side of a polygon having double the number of sides may be determined; and by like repetition, a series of inscribed regular polygons of diminishing sides and increasing perimeters, may be continued to any extent at pleasure.

To exemplify this remark, let the radius $AO=1$, and let $AB=AO$, so that $ABCD$ may be a regular *hexagon.* Then $AD=2$, and in this case $AB=1$, $BD=\sqrt{3}$, the side of an *equilateral triangle.* Let $AB'=B'B$, join $B'D$, and observe that the exterior chord which corresponds to the sides AB', $B'D$ is BD. The formulæ under consideration become $AB'^2=2-BD$, $B'D^2=2+BD$, which may be briefly written,

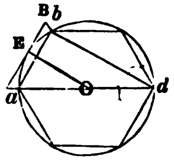
$$s'^2=2-c,\quad c'^2=2+c,$$

and which will make known the side and chord corresponding to a regular inscribed *dodecagon*. In like manner, polygons of twenty-four sides, forty-eight sides, &c. may be determined; the sides and perimeters of which, with those of the hexagon and dodecagon, are given in the following table:

Sides.	*Chords.*	*Perimeters.*
$s=1.0000000$	$c=1.7320508$	$6s=6.0000000$
$s'=0.5176381$	$c'=1.9318516$	$12s'=6.2116571$
$s''=0.2610524$	$c''=1.9828897$	$24s''=6.2652572$
$s'''=0.1308062$	$c'''=1.9957178$	$48s'''=6.2787004$
$s^{\text{iv}}=0.0654382$	&c.	$96s^{\text{iv}}=6.2820639$
&c.		&c.

THEOREM IX.

34. *The side of a regular polygon described about a circle, and that of a similar inscribed polygon, are proportional to the diameter and chord, drawn from the extremities of the side of the inscribed polygon to meet in the circumference.*



Let AB, ab be sides of two regular and similar polygons described about and in the circle abd; and let ad, bd be a diameter and chord; $AB : ab :: ad : bd$.

Draw the radius OE to the point E, in which AB touches the circumference, and let AB, ab be parallel. The right angled triangles AEO, abd are similar, and $AE : ab :: EO : bd$, of which twice the first term is equal to AB, and twice the third equal to ad; wherefore $AB : ab :: ad : bd$.

Remark. If the radius $aO=1$, and the sides AB, ab, and chord bd be denoted by S, s, and c, the proportion becomes $S : s :: 2 : c$, from which and the numerical results of the preceding remark, the sides and perimeters of regular circumscribed polygons having six, twelve, &c. sides, may be determined; and will result as follows:

Sides.	*Perimeters.*
$S =1.1547006$	$6\,S =6.9282036$,
$S' =0.5358984$	$12\,S' =6.4307806$,
$S'' =0.2633050$	$24\,S'' =6.3193200$,
$S''' =0.1310869$	$48\,S''' =6.2921724$,
$S^{\text{iv}} =0.0654732$	$96\,S^{\text{iv}} =6.2854292$,
&c.	&c.

THEOREM X.

35. *The sides and perimeters of regular and similar polygons inscribed in circles and described about them, are proportional to the radii of the corresponding circles.*

For the polygons being similar, their component isosceles triangles formed by joining the angular points and the centres of the circles, are equiangular; and the sides and perimeters of the polygons are therefore proportional to the corresponding radii of the circles.

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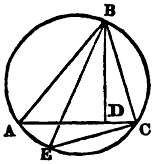
§ 3. OF THE AREAS OF INSCRIBED AND CIRCUMSCRIBED FIGURES.

THEOREM XI.

36. *If a triangle be inscribed in a circle, the product of twice its area, and the diameter, is equal to the product of its sides.*



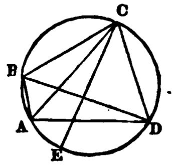
Let A B C be a triangle inscribed in a circle, of which the diameter is B E, twice A B C.B E=A B.B C.C A.

For if B D be the perpendicular to the base; A B.B C= B D.B E (29), which multiplied by A C gives A B.B C.C A= A C.B D.B E. But A C.B D=twice A B C (cor. 85, I); therefore twice A B C.B E=A B.B C.C A.

Remark. The diameter may be found by means of this theorem, when the sides of the triangle are given, since the area in terms of these sides has been investigated in the remark to theorem xii. chapter I.

THEOREM XII.

37. *If a quadrilateral be inscribed in a circle, the product of twice its area and the diameter is equal to the product of either diagonal, and the sum of the products of the sides which diverge from the extremities of the other diagonal.*



For if C E be the diameter, twice A B C.C E=A C.A B.B C, and twice A D C.C E=A C.A D.D C (36); therefore twice A B C E.C E=A C.(A B.B C+A D.D C). In like manner, twice A B C D.C E=B D.(B A.A D+B C.C D).

Corollary. If these be briefly written 2 A B C D.C E= B C.M, and 2 A B C D.C E=B D.L, their comparison will prove theorem vi; and observing that A C.B D=N (cor. 31); there will result 2 A B C D.C E=$\surd$(L.M.N).

Remark. If the sides A B, B C, C D, D A of the quadrilateral be denoted by a, b, c, d; and the diagonal A C by e, there will result 2 A B C D.C E=$e\,(ab+cd)$; and by the last theorem 2 A B C.C E=eab. These by comparison give A B C D : A B C :: $ab+cd : ab$, and thus the area of the quadrilateral A B C D may be determined by the triangle A B C. Now by the remark to theorem xii in the preceding chapter, A B C=$\surd\{p.(p-a)\,(p-b)\,(p-e)\}$, in which $p=\frac{1}{2}(a+b+e)$; and which by multiplication of sums and differences $(a+b+e)$, $a+b-e$; and $e+a-b$, $e-(a-b)$, may be written in the form $ABC=\frac{1}{4}\sqrt{\{[(a+b)^2-e^2][e^2-(a-b)^2]\}}$; and since $AC^2.M=L.N$ (cor. 31), or $e^2(ab+cd)=(ad+bc)(ac+bd)$, this becomes on substituting for e^2 its value, $ABC=\frac{ab}{4(ab+cd)}\sqrt{\{[(a+b)^2-(c-d)^2]\ [(c+d)^2-(a-b)^2]\}}$. The differences of squares in this may be resolved into sums and differences; and putting $p=\frac{1}{2}(a+b+c+d)$, there will result $ABC=\frac{ab}{ab+cd}\sqrt{\{(p-a)(p-b)(p-c)(p-d)\}}$; and thence $ABCD=\sqrt{\{(p-a)(p-b)(p-c)(p-d)\}}$.

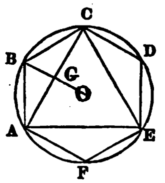
This elegant formula may be thus expressed:

The area of a quadrilateral inscribed in a circle is equal to the square root of the product of the excesses of its semiperimeter above its sides. It involves the area of the triangle as a particular case, when any of the sides is supposed to be nothing; and becomes $ABCD=(p-a)\sqrt{\{(p-c)(p-d)\}}$, when any two sides as BA, AD are equal.

It is evident that the area of the quadrilateral is independent of the arrangement of its sides in the circle, the factors $p-a$, $p-b$, &c. remaining the same whatever be the order in which these sides are connected.

THEOREM XIII.

38. *If a regular polygon and a rectilinear figure of half the number of sides be inscribed in a circle, the area of the polygon is equal to the product of the semiperimeter of the figure and the radius of the circle.*



Let ABC &c. be a regular hexagon, and ACE an equilateral triangle inscribed in the circle BDF, of which the centre is O; the area of A B C &c. is equal to the product of the semiperimeter of A C E and the radius O B.

The polygon A B C &c. is evidently equal to the figure A C E together with as many equal isosceles triangles A B C, C D E, E F A as this figure has sides. But the area of the figure A C E is equal to the product of its semiperimeter p and the perpendicular O G to any side A C; and the total area of the triangles A B C, C D E, E F A is equal to the product of the same semiperimeter p and the perpendicular B G, from any vertex B to the opposite side A C; wherefore the area of the polygon A B C &c. is equal to the product of p and the two perpendiculars O G, G B; that is equal to the product of the semiperimeter of A C E and the radius O B.

Corollary. The radius O B being assumed equal to unity, the areas of the hexagon, dodecagon, &c. may be immediately obtained from the tabulated values of the perimeters given in the remark to (33).

Remark. If a regular figure similar to A C E be described about the circle B D F, and its semiperimeter be denoted by p', its area will be p'.O B, the product of which and the area of the inscribed figure is pp'.O G.O B. But $p' : p :: \text{OB} : \text{OG}$ (34); therefore $p'.\text{OG} = p.\text{OB}$; so that *the area of* A B C &c. *is equal to the product of the semiperimeter* p' *of the circumscribed figure and the altitude* O G *of the similar inscribed figure;* and by virtue of which the preceding product becomes $p^2.\text{OB}^2$. This is the square of p.O B corresponding to the area of A B C &c., *which is therefore a mean proportional between the similar inscribed and circumscribed figures of half the number of sides.*

THEOREM XIV.

39. *The areas of regular and similar polygons inscribed in circles, or described about them, are proportional to the squares of the corresponding radii of those circles, or of their diameters.*

The demonstration of this theorem immediately follows from (35) and (100, I).

THEOREM XV.

40. *The area of a regular polygon inscribed in a circle is equal to the product of its semiperimeter, and a perpendicular drawn from the centre of the circle to any of its sides.*

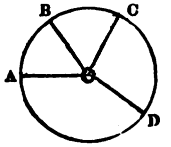
This is evident from (101, I), and the definition of the circle.

41. As to rectilinear figures described about circles, it will be sufficient to observe, that their areas are equal to the products of their semiperimeters and the radii of the circles; the comparative lengths of their sides and their regularity or irregularity being evidently of no consideration.

CHAPTER III.

OF THE ARCS, CIRCUMFERENCES, AND AREAS OF CIRCLES.

42. Two arcs of any circle or of any two equal circles, may evidently subtend equal or unequal angles, formed by radii at their extremities. In the first case, the arcs become identical when the radii forming the equal angles are coincident. In the second case, the greater arc determined by the greater chord, evidently subtends the greater angle at the centre.



In the circle A C D of which O is the centre, A B, B C ar equal arcs subtending the equal angles A O B, B O C; and B C, C D are unequal arcs subtending the unequal angles B O C, C O D, the second of which is the greater angle subtended by the greater corresponding arc C D.

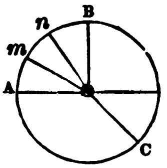
The converse respecting equal or unequal angles relative to equal or unequal arcs is also true; and the like may be said of equal or unequal segments and sectors.

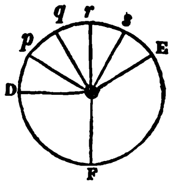
§ 1. OF THE PROPORTION OF ARCS AND ANGLES IN CIRCLES.

THEOREM I.

43. *The arcs of equal circles are proportional to the angles which they subtend at the centres.*

Let A B C, D E F be equal circles in which the arcs AB, D E subtend the angles A O B, D O E at the centres; A B : DE :: A O B : D O E.





For if the angle A O B contains the unit or angle A O *m* three times, and the angle D O E contains the same unit five times, so that A O B : D O E :: 3 : 5; and if the radii forming the equal component angles terminate in the points *m*, *n*, and *p*, *q*, *r*, *s*; the arcs A B, D E will be divided into three and five equal arcs respectively (41); wherefore A B : D E :: 3 : 5. But by hypothesis A O B : D O E :: 3 : 5; and therefore A B : D E :: A O B : D O E.

Corollary. If D O F be a right angle, the corresponding arc D F will be a quadrant, or the fourth part of the circumference, and there will result A B : the quadrant D F :: A O B : the right angle D O F, in which proportion the second term or quadrant D F determines the fourth term or right angle D O F; and these are invariable whatever be the arc A B or the angle A O B. The quadrant D F may therefore be assumed as the measure of the right angle D O F; and according to such assumption, any arc A B of the circle A B C will become the corresponding measure of its subtended angle A O B.

Remark. From the preceding connexion established between any arc of a circle and its subtended angle at the centre, it follows conformably to theorems III. and VI., that an angle at the circumference is measured by half the arc which subtends it; and that the angle formed by two chords or two secants, is measured by half the sum or half the difference of the intercepted arcs.

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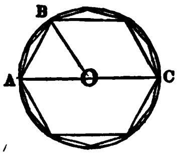
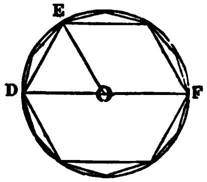
§ 2. OF THE PROPORTION OF THE CIRCUMFERENCES OF CIRCLES.

THEOREM II.

44. *The circumferences of circles are proportional to their corresponding radii, or diameters.*





Let A B C, D E F be circles of which A O, D O are the radii, and A C A, D F D the circumferences; A C A : D F D :: A O : D O.

For if regular and similar polygons be inscribed in the circles A B C, D E F, their perimeters will be proportional to the radii A O, D O (35), or to the diameters A C, D F; and this proportionality will subsist, whatever be the number of sides bounding the polygons. But if this number be continually increased, the corresponding perimeters will also increase, and will continually approach the circumferences A C A, D F D, which form their respective limits, and which must have their invariable ratio A O : D O. Wherefore the circumference A C A : cirumference D F D :: A O : D O.

Corollary. Similar arcs of circles, subtending equal angles at the centres, are proportional to the circumferences, and therefore proportional to the corresponding radii or diameters.

THEOREM III.

45. *The circumference of a circle is to the diameter nearly as* 3.1415926 *to* 1.

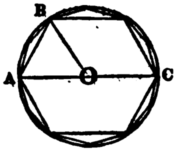
For if regular and similar polygons of 12288 sides be inscribed in and described about the circle, their perimeters, determined by the remarks to theorems IX. X. of the preceding chapter, will result 6.2831852 and 6.2831854, corresponding to radius 1. But the circumference of the circle is evidently greater than the first of these perimeters, and less than the second; and must be nearly equal to 6.2831853, a mean between them; wherefore the circumference is to the diameter nearly as this mean is to 2; or as 3.1415926 to 1.

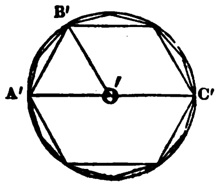
Remark. The ratio of the circumference of a circle to the diameter is usually denoted by the letter π, which is said to express the semicircumference corresponding to radius 1. Instead of its approximate value here given, the number 3.1416 and the fractions $\frac{22}{7}$, $\frac{355}{113}$ are frequently employed. These correspond to inscribed and circumscribed polygons of 768, 96, and 3072 sides. The number 3.1416 is equal to the fraction $\frac{3927}{1250}$, which is said to be known to the Bramins of India. The fraction $\frac{22}{7}$ was found by Archimedes, and $\frac{355}{113}$ is ascribed to Metius, a German mathematician of the sixteenth century. It is evident that by increasing the number of sides of polygons described in and about a circle, their perimeters will continually approach the circumference; and that the length of this circumference may be considered as numerically determined within any assigned degree of accuracy.

§ 3. OF THE PROPORTION OF THE AREAS OF CIRCLES.

THEOREM IV.

46. *The areas of circles are proportional to the squares of their radii, or of their diameters.*





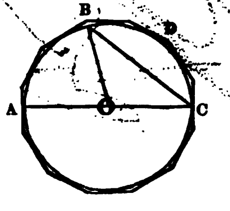
Let A B C, A′ B′ C′ be circles of which the radii are A O, A′O′; the area of A B C : the area of A′ B′ C′ :: A O^2 : A′ O'^2.

For if regular and similar polygons be inscribed in the circles A B C, A′ B′ C′, their areas will be proportional to the squares of the radii A O, A′ O′ (39), or to the squares of the diameters A C, A′ C′, whatever be the number of their sides, and however near they may approach the circles A B C, A′ B′ C′, which form their respective limits, and which must have their invariable ratio A O^2 : A′ O'^2. Wherefore the area of A B C : the area of A′B′C′ :: A O^2 : A′O'^2.

Corollary. The areas of the similar sectors A O B, A′O′B′, having equal angles at O, O′, are evidently proportional to the circles A B C, A′B′C′; and are therefore proportional to the squares of the radii A O, A′O′, or of the diameters A C, A′C′.

THEOREM V.

47. *The area of a circle is equal to the product of its semicircumference and radius.*



Let A B C be a circle of which the radius is A O; the area of A B C is equal to the product of the semicircumference A B C and radius A O.

For if a regular polygon be described about the circle A B C, its area will be equal to the product of its semiperimeter and the radius A O; and this area and corresponding product, by increasing the number of sides of the polygon, will continually approach the area of the circle and the product of its semicircumference and radius, which form their respective limits, and between which there must subsist a corresponding equality. Wherefore the area of the circle

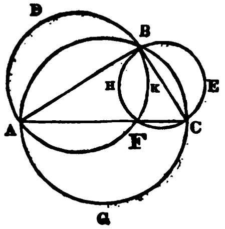
A B C is equal to the product of its semicircumference A B C and radius A O.

Corollary. The area of the sector A O B is equal to the product of its semiarc A B and the radius A O.

Remark. The segment B C D B is evidently equal to the difference between the corresponding sector O B D C, and the isosceles triangle B O C. The area of this segment will therefore result from the areas of the corresponding sector and triangle, determined by the preceding corollary and article 122, I.

THEOREM VI.

48. *The area of a circle described on the hypotenuse of a right angled triangle, as a diameter, is equal to the areas of circles, similarly described on the other sides.*



Let A B C be a right angled triangle of which A C is the hypotenuse; the circle A B C is equal to the circles A D B, B E C.

Because circles are proportional to the squares of their diameters, the circle A D B : circle B E C :: $AB^2 : BC^2$; and by composition of ratios, the circle A D B : circle A D B + circle B E C :: $AB^2 : AB^2 + BC^2$. But $AC^2 = AB^2 + BC^2$ (54, I), and the circle A D B : circle A B C :: $AB^2 : AC^2$; wherefore by comparison of equal ratios, the circle A B C = the circle A D B + circle B E C.

Corollary. The semicircle A B C being equal to the semicircles A D B, B E C, *the triangle* A B C is equal to *the lunes* A D B A, B E C B; and the semicircle A G C being equal to the semicircles A F B, C F B, *the triangle* A B C is equal to the difference of *the cusped space* A F C G and *the lenticular space* B H F K; so that *the cusped space* A F C G is equal to *the two lunes* A D B A, B E C B, and *the lenticular space* B H F K.

CHAPTER IV.

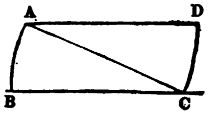
OF CONSTRUCTIONS AND MENSURATION RELATIVE TO CIRCLES.

49. THE problems contained in this chapter complete practical geometry and mensuration, as far as regards elementary plane figures. The construction of some of them already considered in the preceding book, are here effected in the usual mode, by the combined assistance of the circle and the right line. A number of them are also resolved independently of the right line, that the student may not be unapprized of some ingenious resources left to a geometer, limited to a compasses, as the only mechanical instrument of solution.

§ 1. OF CONSTRUCTIONS RELATIVE TO THE CIRCLE.

PROBLEM I.

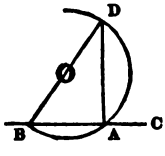
50. *Through a given point* A *to draw a right line parallel to a given right line* B C.

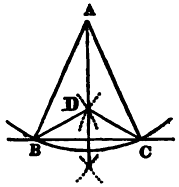


From A as a centre describe the arc C D; and from the centre C describe the arc A B. Take C D equal to A B, and join A D which will evidently be parallel to C D.

PROBLEM II.

51. *From a given point* A *to draw a perpendicular to a given right line* B C.





First, let A be in the line B C, and from O any point without B C, describe the semicircle B A D; join A D, which will be perpendicular to B C (12).

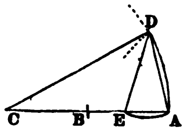
The point D may be determined by applying the radius O A as a chord three times round the arc from B to D.

Secondly, let A be without B C; and from it as a centre describe the arc B C. From B and C as centres describe arcs intersecting in D on either side of B C. Draw A D to meet B C perpendicularly as required.

For the triangle A B C is isosceles; and the triangles A B D, A C D having equal sides, are identical: the angle B A D is therefore equal to C A D, and A D is perpendicular to B C (cor. 33, I).

PROBLEM III.

52. *To divide a given right line* A B *into two or more equal parts.*



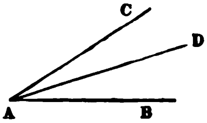
In the direction of A B, take A C equal to A B, repeated twice or oftener, according to the number of parts required. From C as a centre with the radius C A, describe the arc A D, and take its chord A D equal to A B. From D, with the radius A B, describe an arc intersecting A B in E; A E is the required part of A B

For the similar isosceles triangles A C D, A D E give A C : A D :: A D : A E, in which A C is twice A D or A B, and therefore A B=2 A E. In like manner, there will result A B=3 A F, A B=4 A G, &c. when A C=3 A B, A C= 4 A B, &c.

Remark. The usual construction has been already given in the first Book, Chap. V.

PROBLEM IV.

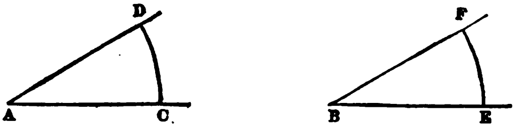
53. *To divide a given angle* A *into two equal parts.*



In the sides of the given angle A, take equal parts A B, A C; and from B, C as centres describe arcs intersecting in D. Join A D, which will evidently bisect the angle A.

PROBLEM V.

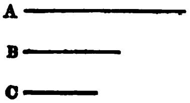
54. *At a given point* A *to make an angle equal to a given angle* B.

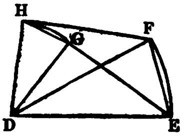


From A, B as centres describe arcs C D, E F, the latter intersecting the sides of B in E, F. Take C D equal to E F; and join A C, A D which will evidently form the angle A equal to B.

PROBLEM VI.

55. *To find a fourth proportional to three given right lines* A, B, C.





From any point D with radii A, B describe arcs E F, G H, and let the chord E F be equal to C. From E with the distance F H describe an arc intersecting G H in G; the chord G H is the fourth proportional required.

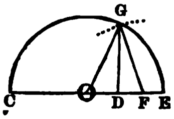
For the triangles D E G, D F H are identical, and the angle E D G is equal to F D H. From these angles take F D G, and the angle E D F is equal to G D H. Wherefore the isosceles triangles E D F, G D H are similar, and D E : D G :: E F : G H, or A : B :: C : G H, the required proportional.

Remark. When B$=$A, the chord G H is a third proportional to the two lines A and C.

As to the usual construction of this problem, see 108. I.

PROBLEM VII.

56. *To find a mean proportional between two given right lines* A, B.

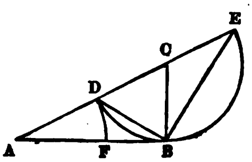


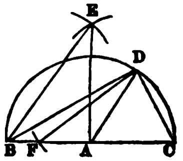
Let A and B constitute the right line C D E, which bisect in O. Take D F equal to D O, and from O describe the semicircle C G E. From F with the radius O G describe an arc intersecting the semicircumference in G. Join D G, which will be the required proportional.

For D G is evidently perpendicular to the diameter C E; and $CD, DE = DG^2$ (rem. 16); or $A.B = DG^2$.

PROBLEM VIII.

57. *To divide a given right line* A B *in extreme and mean ratio, or so that the square of the greater part may be equal to the product of the whole line and the less part.*





At B erect the perpendicular B C equal to the half of A B, and through A, C draw A C E. From C with the radius C B describe the semicircle D B E, and from A with the radius A D describe the arc D F. The point F will divide A B as required (rem. 17).

Corollary. The secant A E is divided in extreme and mean ratio at D; since $AB^2 = AD.AE$, and $AB = DE$. It also appears from the similar triangles A D B, A B E in which A D or A F : A B :: B D : B E, that B D is the greater part of B E, also divided in extreme and mean ratio; and that DE^2 or $AB^2 = BE^2 + BD^2$, or the square of B E and the square of its greater segment, formed by dividing it in extreme and mean ratio.

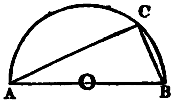
Remark. This problem may be resolved by a compasses thus: From A describe the semicircle B D C, determined by applying A B as a chord twice from B to D, and once from D to C. From B, C with the radius B D describe arcs at E; and from D with the radius A E describe an arc, intersecting A B in F, the point required.

For the triangle A D C being isosceles, $FD^2 = AD^2 + FA.FC$, or $AE^2 = AB^2 + FA.FC$; and adding AB^2, there results $BE^2 = 2\,AB^2 + FA.FC$. But BE^2 or $BD^2 = BC^2 - CD^2 = 3\,AB^2$; wherefore $AB^2 = FA.FC$, from which subtracting $AB.FA$, there remains $AB.BF = FA^2$.

Corollary. It appears from $AB^2 = FA.FC$, that F C is divided in extreme and mean ratio at the point A.

PROBLEM IX.

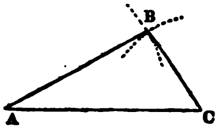
58. *To construct a right angled triangle of which the hypotenuse* A B *and the side* A C *are given.*



On A B as a diameter describe the semicircle A C B, and from A with the radius A C, describe an arc at C. Join A C, B C, which will evidently be the required triangle.

PROBLEM X.

59. *To construct a triangle of which the sides* A B, B C, C A *are given.*

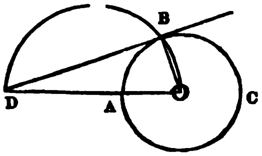


From the extremities of the base A C, with radii equal to the sides A B, B C, describe arcs at B. Join A B, B C. The triangle A B C is evidently that required.

Remark. For the construction of a triangle, when two sides and the included angle, or when one side and the adjacent angle are given, see Chap. V., B. I.

PROBLEM XI.

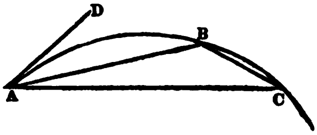
60. *To draw a tangent to a given circle* A B C, *so that it may pass through an exterior given point* D.



Join D and the centre O, and on D O describe the semicircle D B O, cutting the circle A B C in B. Join D B, which will be perpendicular to B O, and therefore a tangent at B (cor. 9).

PROBLEM XII.

61. *To draw a tangent to a given circular arc* A B C *from one of its extremities.*



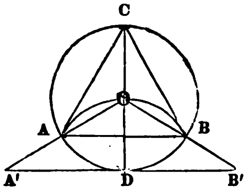
Draw any two chords A B, B C; join A C, and at the point A draw A D, making the angle B A D equal to A C B (54); the right line A D is a tangent to the arc A B.

For the angle B A D, formed by the chord A B and right line A D, being equal to the angle A C B in the alternate segment A B C, this right line is evidently a tangent to the arc A B.

Remark. The chords A B, B C may be taken equal to each other, in which case the angle B A D will be equal to B A C; and corresponding arcs described from A with the radius A B, will determine the position of A D.

PROBLEM XIII.

62. *To inscribe an equilateral triangle in a given circle* A B C.



From any point D in the circumference, describe an arc A O B through the centre O of the circle; the chord A B is the side of the required triangle.

For the chord A B evidently corresponds to a third part of the circumference, and is therefore the side of the inscribed equilateral triangle.

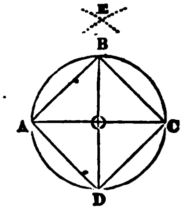
Remark. If a tangent be drawn at D, and the radii A O, O B be produced to meet it in A′, B′; the side of the circumscribing equilateral triangle will be A′B′; and the radius of a circle in which it may be inscribed is O A′. This side and radius are respectively equal to 2 A B and 2 O A, since O D is bisected by the chord A B.

In like manner, may the side of any rectilinear figure circumscribing a circle be determined from the side of a similar inscribed figure; a remark which is applicable in case of the following and other figures, and need not be repeated.

Should the side A B of the equilateral triangle be given, the figure may obviously be constructed by arcs described from A and B with the radius A B, as in problem x.

PROBLEM XIV.

63. *To inscribe a square in a given circle* A B C.



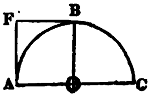
Draw the diameters A C, B D perpendicular to each other, and join A B C D, which is evidently the required square.

Remark. The point C may be determined by applying A O as a chord three times round the circumference; and the point E may be found as in problem VIII. The distance O E applied as a chord from A, will determine the points B and D; and thus the angular points of the required square will become known.

To prove that O E is equal to the diagonal A B; the right angled triangle A O E gives $OE^2 = AE^2 - AO^2 = 2AO^2$, and the triangle A O B also gives $AB^2 = 2AO^2$; wherefore O E is equal to A B, and the construction is obvious.

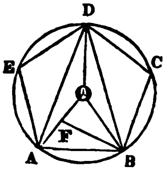
Should the side A O of a square be given, the point B may be determined in the manner just explained; and the remaining angular point F will be the intersection of arcs described from A and B with the radius A O.

With regard to the circumscribing square, see the remark to the preceding problem, or draw parallels to A C and B D at B, D and A, C.



PROBLEM XV.

64. *To inscribe a regular pentagon in a given circle* A B C.



Divide the radius O A in extreme and mean ratio, and let the greater segment be O F. From F as a centre with the radius O A, describe an arc, cutting the circle A B C in B. Join A B, which is the side of the required pentagon A B C D E.

For join B O, B F; draw the equal chords A D, B D, and the radius O D. Because the base O F of the isosceles triangle F B O is equal to the greater segment of its side B F, divided in extreme and mean ratio, the angle B O F is twice the angle O B F (cor. 57, I). But the angle B O F is also twice the angle A D B (11); wherefore the isosceles triangles F B O, A D B are equiangular, and the angle B O F or A O B is equal to A D B or B A D. Each of the angles A O D, B O D is therefore twice the angle A O B; and the right line A B, subtending a fifth of the angles at the centre O, is the side of the required pentagon.

Remark. The triangle F B O being isosceles, $AB^2 = OB^2 + OA.AF = OA^2 + OF^2$, in which the radius O A is equal to the side of a regular inscribed hexagon (95, I), and the segment O F equal to the side of a regular inscribed decagon (rem. 32). This formula may therefore be expressed in the following terms:

The square of the side of a regular pentagon inscribed in a circle, is equal to the square of the side of a regular inscribed hexagon, together with the square of the side of a regular inscribed decagon.

It will also be proper to observe, that if the side A B be given, the chord A C may be formed from it, precisely as

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A E has been determined in problem VIII. The corresponding pentagon may then be constructed by the radii A B, A C, and arcs described from the points A and B.

PROBLEM XVI.

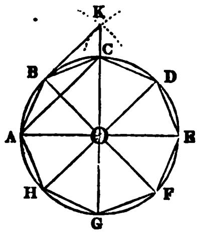
65. *To inscribe a regular hexagon in a given circle* A B C.

The construction of this figure is obvious: The radius applied as a chord six times round the circumference will determine the angular points of the hexagon.

Remark. If the side be given, an equilateral triangle described on it will determine the centre of a circle in which the angular points of the figure may be found, in the manner just mentioned.

PROBLEM XVII.

66. *To inscribe a regular octagon in a given circle* A B C.

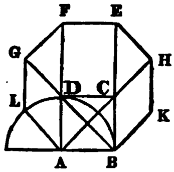


Draw the diameters A E, C G perpendicular to each other. Apply the diagonal A C from the centre O to K, in the prolongation of O C. From K as a centre with the radius A O, determine the points B, D; by which F, H will become known. Join the several points A B C &c., which will form the required octagon.

For the isosceles triangles O A C, B O K are identical; and the angle B O C being equal to C A O, is half a right angle. Wherefore the point B is in the required octagon.

Remark. The points A, C, E, G may be determined as the angular points of the square in prob. XIV.; and K may also be found in the same manner as the point E, prob. VIII.

Should the side of the octagon be given, the centre of the circumscribing circle may be found by problem XII., B. I.; or the following construction may be adopted:



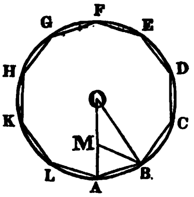
On A B, the given side, describe the square A B C D; and let the prolongations of its sides A D, B C, and of its diagonals A C, B D be respectively equal to the diagonal A C and side A B; by which the points E, F, G, H will be determined. The remaining points K, L may be found by arcs described from B, H and A, G, with the radius A B.

For the several angles are evidently formed of a right angle, and half a right angle equal to B A C; wherefore the figure is a regular octagon.

If a semicircle be described from A with the radius A B, and a chord be drawn as A F in problem XI., this chord will be equal to D E or C F. The points E, F may therefore be determined by a compasses; and these with the four points already known, will determine, by the intersection of arcs described with the radius A B, the remaining vertices of the octagon.

PROBLEM XVIII.

67. *To construct a regular decagon in a given circle* A B C.



Divide the radius O A in extreme and mean ratio, and let O M be the greater segment. Apply O M ten times round the circumference, and the several angular points of the required decagon will be determined (rem. 32).

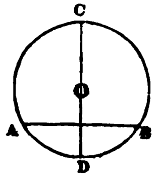
Remark. Should the side A B of the decagon be given, the radius A Θ may be found as A E in problem VIII.

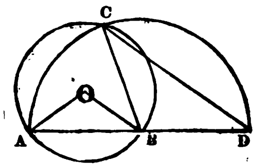
The polygons constructed in the preceding problems are those usually considered in elementary geometry. From the hexagon other polygons may be obtained by continually bisecting the angles at the centre of the circles; and the like may be said relatively to the octagon and decagon, which will give polygons of 16, 32, 64, &c., and 20, 40, 80, &c., sides respectively. A third series of polygons may be formed by inscribing a regular hexagon and decagon, so that they may have two of their vertices at the extremities of the same diameter. The arcs between other adjacent vertices will then be $\frac{1}{15}$ or $\frac{1}{30}$ of the circumference, corresponding to polygons of 15 and 30 sides, and from which polygons of 60, 120, 240, &c. sides may be derived by bisections of angles, as in case of the hexagon, octagon, or decagon.

In addition to the polygons here mentioned, and which were known to ancient geometers, Mr. Gaus, a distinguished German mathematician, has proved in his *Disquisitiones Arithmeticæ*, published in 1801, that polygons having 17 sides, 257 sides, and in general polygons, of which the number of sides is expressed by a prime number of the form 2^n+1, may be geometrically constructed. The mode of inscribing these figures in circles is extremely though necessarily complex, as may be readily imagined, and depends on considerations in advance of the present treatise.

PROBLEM XIX.

68. *To find the centre of a given circle* A B C.





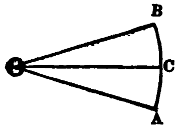
Draw the chord A B, which bisect perpendicularly by the diameter C D. Bisect C D in O, which is evidently the required centre.

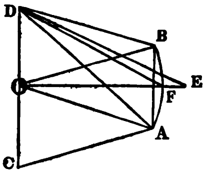
Remark. To determine O independently of the right line, from B describe the semicircle A C D, cutting the circumference in the points A and C. Find A O a third proportional to C D and A B, which will be equal to the radius of the given circle, and by which the centre O may be immediately determined.

For O being the centre, the isosceles triangles A O B, C B D are evidently similar, and C D : B C or A B :: A B : A O, which is therefore a third proportional to C D, A B, and leads to the preceding construction.

PROBLEM XX.

69. *To bisect a given arc* A B *of a circle, of which the centre* O *is given.*





Draw the radii O A, O B, and bisect the angle A O B by the right line O C, which will evidently bisect the arc A B.

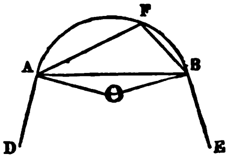
Remark. The arc A B may also be bisected by means of a compasses. From A, O with the radii A O, A B describe arcs intersecting at C, so that O C may be parallel to A B; and from B, O determine in like manner the opposite point D. From C, D with the radius A D describe arcs at E; and from either of the same points with the radius O E, describe an arc cutting A B in F, the point required.

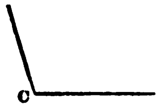
For A O C is an isosceles triangle, and A D terminates in D, a point in the prolongation of its base; wherefore $DA^2 = AO^2 + DO.DC = AO^2 + 2DO^2$; or $DA^2 = FO^2 + 2DO^2$. But F O and D O being perpendicular to each other, $DF^2 = FO^2 + DO^2$; and therefore $DA^2 = DF^2 + DO^2 =$

$E O^2 + D O^2$ or $D A^2 = D E^2$, from which $D A = D E$; and the construction is obvious.

PROBLEM XXI.

70. *To describe on a given right line* A B, *a segment of a circle, capable of containing a given angle* C.





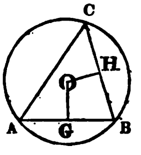
Draw A D, B E, making the angles B A D, A B E, each equal to C; and perpendicularly to these lines draw A O, B O, which will intersect in O, the centre of the circle, determining the required segment.

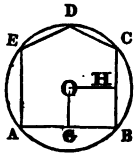
For A D, B E forming with the chord A B angles respectively equal to the angle F in the segment A F B, are tangents to the circumference at A, B (13); and therefore A O, B O are radii.

Remark. The radius A O may be determined by means of a semicircle described from B and by a third proportional, as in problem XIX.

PROBLEM XXII.

71. *To describe a circle about a given triangle* A B C, *or about any rectilinear figure, when such is possible.*

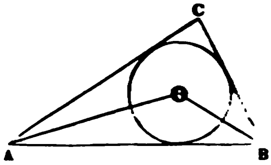


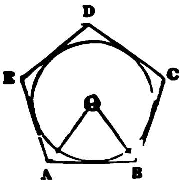


Bisect any two adjacent sides A B, B C by perpendiculars GO, H O, which will evidently determine the centre O.

PROBLEM XXIII.

72. *To inscribe a circle in a given triangle* A B C, *or in any rectilinear figure, when such is possible.*





Bisect any two angles A, B, adjacent to the same side AB. The point O in which the bisecting lines meet, will evidently be the centre of the circle.

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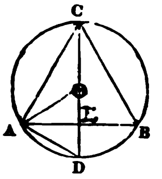
§ 2. OF MENSURATION RELATIVE TO THE CIRCLE.

PROBLEM I.

73. *To determine the side of an equilateral triangle inscribed in a given circle.*



The side$=r.\sqrt{3}$, ... [1]

in which r is the radius A O.

For draw the diameter C D perpendicular to A B. The chord A D is equal to the radius A O, and therefore $AC^2 = CD^2 - AO^2 = 3\,AO^2$, from which $AC = r\sqrt{3}$.

Corollary. Since O E is the half of O D, the altitude $CE = \frac{3}{2}\,r$.

Example. Let $r=10$ yards; then $AB = 10\sqrt{3} = 17.3205$, and $CE = 15$ yards.

Remark. The side of a circumscribing equilateral triangle is twice that of the inscribed figure (rem. 62).

PROBLEM II.

74. *To find the area of an equilateral triangle inscribed in a given circle.*

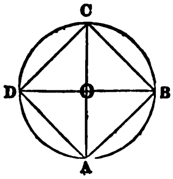
The area$=\frac{3}{4}\sqrt{3}.r^2$ . . . . [2].

For the base is $r\sqrt{3}$, and the altitude $\frac{3}{2}r$, and therefore the area is expressed by formula [2].

Example. If $r=10$, the area$=129.9037$.

PROBLEM III.

75. *To determine the side of a square inscribed in a given circle.*



The side$=r\sqrt{2}$. . . . . [3].

For $AB^2=AO^2+OB^2=2\,AO^2$, and therefore $AB=r\sqrt{2}$.

Remark. The side of the circumscribing square is obviously equal to the diameter.

PROBLEM IV.

76. *To find the area of a square inscribed in a given circle.*

The area$=2\,r^2$. . . . . [4].

For the side$=r\sqrt{2}$, the square of which is $2\,r^2$.

Example. Let $r=12$ feet, 9 inches. The area$=325$ feet, $1\frac{1}{2}$ inches, square measure.

PROBLEM V.

77. *To find the area of a quadrilateral inscribed in a given circle.*

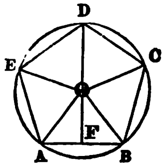
The area$=\sqrt{\{(p-a)\,(p-b)\,(p-c)\,(p-d)\}}$ . . [5],

in which p is the semiperimeter, and a, b, c, d, the sides of the inscribed quadrilateral (rem. 37).

Example. Let $a=3$, $b=4$, $c=5$, $d=6$. Then $p=9$, $p-a=6$, $p-b=5$, $p-c=4$, $p-d=3$, and the area$=18.97$.

PROBLEM VI.

78. ***To determine the side of a regular pentagon inscribed in a given circle.***



The side$=r\surd\{\frac{5-\surd5}{2}\}$ . . . . [6].

For if the radius A O$=r$ be divided in extreme and mean ratio, by erecting a perpendicular at A equal to its half, as in prob. VIII.; the greater segment will be the difference between $\surd(r^2+\frac{1}{4}r^2)$ and $\frac{r}{2}$, and which is $\frac{r}{2}(\surd5-1)$. But the square of A B is equal to the square of this segment and that of A O; wherefore $AB^2=\frac{r^2}{4}(6-2\surd5)+r^2=r^2.\frac{5-\surd5}{2}$, and A B is expressed by formula [6].

Corollary. The altitude $OF=\surd\{AO^2-\frac{1}{4}AB^2\}=\frac{1}{2}r.\surd\{\frac{3+\surd5}{2}\}=\frac{1}{4}r.(\surd5+1)$.

Remark. If AB$=s$, there will result AO or $r=s.\surd\{\frac{5+\surd5}{10}\}$, and O F$=\frac{1}{2}s\surd\{\frac{5+2\surd5}{5}\}$, by virtue of which the radius of a circle, circumscribing a given polygon becomes known, and also the altitude of the polygon.

Example. Let $r=10$ feet; then A B$=11.7556$, and O F$=8.0901$. The side of a circumscribing pentagon$=\frac{AO}{OF}.AB$.

PROBLEM VII.

9. ***To find the area of a regular pentagon inscribed in a given circle.***

$$\textit{The area}=r^2.\tfrac{5}{4}\surd\{\frac{5+\surd5}{2}\}\ldots\ldots.[7].$$

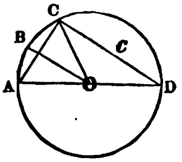
For the side $AB=r\surd\{\frac{5-\surd5}{2}\}$, and the altitude $OF=\frac{1}{2}r\surd\{\frac{3+\surd5}{2}\}$, therefore the area of the component triangle $AOB=\frac{1}{4}r^2\surd\{\frac{5+\surd5}{2}\}$, and that of the pentagon is expressed by formula [7]. (See the preceding figure).

Remark. If $AB=s$, the area$=\frac{1}{4}s^2\surd\{25+10\surd5\}=1{,}720477\ s^2$, from which the area of any regular pentagon of which the side is given may be found.

Example. Let $r=10$, then the area$=237{,}762$. The area of the circumscribing pentagon will result from this multiplied by $\frac{AO^2}{OF^2}$ (100, I).

PROBLEM VIII.

80. *To determine the side of a regular polygon inscribed in a given circle, having the side of an inscribed regular figure of half the number of sides.*



$$\textit{The side } AB=\surd(2r^2-rc)\,;\ldots\ldots[8],$$

in which c is the chord $CD=\surd(4r^2-s^2)$, joining the given side $AC=s$, and the diameter $AD=2r$ (cor. 33).

Corollary. The altitude corresponding to the required side is $\frac{1}{2}\surd(2r^2+rc)$.

Remark. The side AB is also equal to the excess of $\surd(r^2+\frac{1}{2}rs)$ above $\surd(r^2-\frac{1}{2}rs)$; and the corresponding altitude is half the sum of these roots, found by extraction, after the substitution of $\surd(4r^2-s^2)$ instead of c. From the formula [8], and the known sides of an inscribed equilateral triangle, square, and pentagon, there results for the sides of the hexagon, octagon, decagon, and dodecagon, the following values: r, $r\sqrt{(2-\sqrt{2})}$, $\frac{1}{2}r(\sqrt{5}-1)$, and $\frac{1}{2}r(\sqrt{6}-\sqrt{2})$; or r; $.76536\,r$; $.61803\,r$; $.51764\,r$; which may be rendered particular by taking any number for r.

PROBLEM IX.

81. *To find the area of a regular polygon inscribed in a given circle, having the side of an inscribed regular figure of half the number of sides.*

The area$=pr$ . . . . [9],

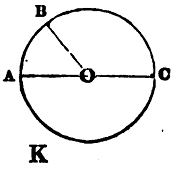
in which p is the semiperimeter, corresponding to the given side s, and r the radius (38).

Example. Let $r=10$, and suppose the polygon corresponding to the given side to be a regular hexagon. The area of the dodecagon will then be $30.10=300$.

Remark. The preceding formula immediately results from the expressions given for the side and altitude in the last problem; for half the product of these is $\frac{r}{2}\sqrt{(4r^2-c^2)}=\frac{1}{4}rs$; which multiplied by the number of sides forming the perimeter $2p$, will give the expression [9]. The area pr is also a mean proportional between the area pm of the inscribed figure, and the area $\frac{pr^2}{m}$, of the similar circumscribing figure, the altitude corresponding to p being represented by m (rem. 38).

PROBLEM X.

82. *To find the circumference of a circle, of which the radius or the diameter is given.*



The circumference $c=2\,r\,\pi$. . . . . . . [10],

in which r is the radius, and $\pi=3.1415926$.

Example. Let the diameter$=20$ yards. The radius $r=10$ and the area$=62.831852$ yards.

Corollary. When the circumference c is given, the diameter $2\,r=\frac{c}{\pi}$.

Remark. Instead of the preceding value of π, the fraction $\frac{355}{113}$ may be used, or either of the values 3.1416, $\frac{22}{7}$, according to the degree of accuracy required.

PROBLEM XI.

83. *To find the area of a circle of which the radius, or the diameter, is given.*

The area$=r^2\,\pi$. . . . . [11].

For the area$=\frac{1}{2}\,c\,r$ (47), and $\frac{1}{2}\,c=r\,\pi$, by the last problem; wherefore the area$=r^2\,\pi$.

Example. Let the diameter $2\,r=20$ feet, then $r=10$ and the area$=314.15926$

PROBLEM XII.

84. *To find the area of a sector of a circle, having the arc and radius given.*

The area$=\frac{1}{2}\,a\,r$. . . . . [12],

in which a denotes the arc of the sector.

Example. Let a be one tenth of the circumference, corresponding to $r=6$ feet. The circumference $c=37.6992$, the arc $a=3.76992$; and the area$=3\times3.76992=11.30976$ square feet.

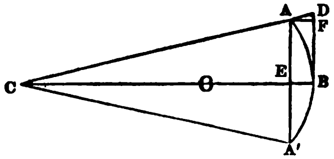
PROBLEM XIII.

85. *To find the arc of a circular segment of which the chord and height are given.*

The arc $a=\frac{c\,m}{m-h}$ *nearly*. . . . . . [13],

in which h is the height of the segment, c the chord, and

$m=3\,r-\frac{1}{4}\,h$, or $3\,r-\frac{1}{5}\,h$, according as h is greater or less than half the radius r.



For let A B A′ be a segment, of which the semiarc A B is one fifth of the circumference described from O with the radius O B=1. Then O E=.309017 being the semichord of a regular inscribed decagon, there will result AE or $\frac{1}{2}c=.951057$, B E or $h=.690983$, and B C or $m=2.827255$. By virtue of these values, the first formula gives $a=2.51738$, which is nearly equal to the true value $a=2.51328$. The same formula in the extreme case of $h=1$ gives $a=\frac{22}{7}$, one of the approximate values of π; and these cases sufficiently show its application and accuracy. As to the second formula, let A A′ be the side of a regular inscribed decagon. Then $c=.618034$, $h=.048943$, $m=2.990211$, and $a=.628318$, which is extremely near the true value of the arc.

Remark. From either of the formula here given, there results $a=c\,(1+\frac{1}{3}\,h)$, when h is considered sufficiently small to neglect its square, and higher powers. According to this supposition, $h=\frac{1}{8}\,c^2$, and therefore $a=c+\frac{1}{24}\,c^3$, which may be employed to determine the arc of a segment of which the height is small relatively to its chord.

PROBLEM XIV.

86. *To find the area of the segment of a circle, of which the radius, chord, and arc are given.*

$$\textit{The area}=\tfrac{1}{2}\,ar-\tfrac{1}{4}\,c\sqrt{(4\,r^2-c^2)}\,\ldots\ldots\,[14],$$

in which c is the chord, and a the arc.

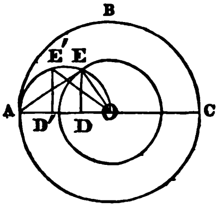
For the area of the corresponding sector is $\frac{1}{2}\,a\,r$, and the altitude of the isosceles triangle, of which the base$=c$, is $\sqrt{(r^2-\frac{1}{4}\,c^2)}$; wherefore the area of this triangle is $\frac{1}{4}\,c\,\sqrt{(4r^2-c^2)}$; and the area of the segment is expressed by formula [14].

Example. Let a be one tenth of the circumference of which the radius is 6 feet. Then c being 3.7082, the area $=11.30976-1.8541\times5.70646=.72972$.

Remark. If the height h and chord c be given, the square of the chord of half the arc will be $c^2+h^2=2rh$ (54, I), from which the radius $r=\frac{c^2+h^2}{2h}$.

PROBLEM XV.

87. *To find the area of a circular annulus or ring, of which the interior and exterior diameters are given.*



The area $=(r^2-r'^2)\pi$. . . . . [15],

in which r and r' are the exterior and interior bounding circumferences.

Example. Let $r=8$ inches, and $r'=6$ inches. The area $=(8^2-6^2)\pi=87.9646$ square inches.

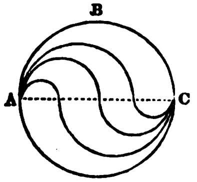
Remark. Should the annulus be equal to any number of times the interior circle, as for instance twice, this circle will be a third of A B C; and therefore $r'^2=\frac{1}{3}r^2$, from which $r'=\frac{r}{3}\sqrt{3}$. This may be determined in the following manner: On A O describe the semicircle A E O. Divide A O into three equal parts, and at D the first division reckoned from the centre O, erect the perpendicular D E, meeting the semicircle in E. The required radius is O E. For $OE^2=AO.DO$ (54, I)$=\frac{1}{3}AO^2$, or $r'^2=\frac{1}{3}r^2$, as above given.

By means of another perpendicular at the middle of A D, a corresponding radius O E' may be determined, so that the circumference described through E' will form with the interior and exterior circumferences annuli equal to each other and to the interior circle; and in a similar manner may A B C be divided into any number of equal parts whatever.

The preceding construction and formula [15] are not limited to annuli formed by concentric circles, but will apply to those bounded by eccentric circles, and even to lunes having their cusps at the point A or C. The division of A B C may therefore be effected independently of the particular situations of the interior circles, of which the radii are O E, O E′, &c.

In addition to the present remark, it may be proper to notice a simple construction by which the circle A B C may be divided into parts, having not only equal surfaces, but also equal perimeters. The views which first led to the possibility of this division, may be seen in Dr. Hutton's *Mathematical Tracts*. The construction is the following:

Divide the diameter A C into as many equal parts as required in the circle A B C. Through each point of division and the extremities A, C, describe semicircles on different sides of A C. The cusped spaces thus enclosed will have equal surfaces and also equal perimeters.



For if R be the radius of A B C, and r, r' the radii of any two consecutive semicircles on the same side of A C, the corresponding radii of the semicircles on the opposite side will be $R-r$, $R-r'$, and the area of the enclosed cusped space will be $R(r-r')\pi$ (83). But A C being divided into a given number of equal parts, as for instance four, $r-r'=\frac{1}{4}R$, and the preceding area becomes $\frac{1}{4}R^2\pi$, which is obviously the fourth part of the circle A B C. As to the perimeter, it is evidently equal to $2R\pi$, the circumference of A B C (82).

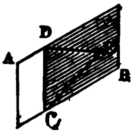
BOOK III.

OF PLANES AND PLANE SOLIDS.

CHAPTER I.

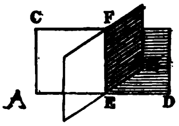
OF THE RELATIVE POSITIONS OF PLANES.

1. A **PLANE** may, in accordance with its definition, pass through a right line joining any two points; and it may evidently be so posited as to pass at the same time through a third point without that line. It follows therefore that a plane may be drawn through any three points whatever; and that any three right lines which meet each other, are in one and the same plane.



The plane A B passes through the three points C, D, E; and also through the three right lines C D, D E, E C, which join these points, and which form the triangle C D E.

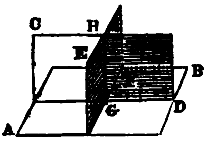
2. Two planes evidently intersect each other in a right line. This line being in both planes is termed their *common section.*



Of the planes A B, C D which cut each other, the common section is the right line E F.

3. Three planes intersect each other in three right lines, of which [illegible] one may evidently be perpendicular to each of

the remaining two. It follows therefore that a right line which meets a plane in a point, may be perpendicular to two other right lines drawn in the plane from that point; and also that any three right lines which meet in a point may be perpendicular to each other.

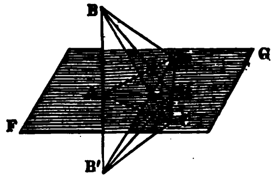


The planes A B, C D, E F intersect in the right lines G D, G F, G H, of which G H may evidently be perpendicular to G D, G F in the plane A B; and these two lines G D, G F, by turning the plane C D about the line G H, may evidently be perpendicular to each other.

§ 1. OF THE POSITIONS OF RIGHT LINES RELATIVELY TO A PLANE.

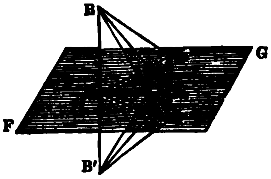
THEOREM I.

4. *If a right line be perpendicular to each of two right lines at a common point, it is perpendicular to every other right line in their plane at that point.*



Let the right line A B be perpendicular to each of the lines A C, A D, in the plane F G; and let A E be any other right line in this plane; A B is perpendicular to A E.

In B A produced, take A B′ equal to A B; join C D, meeting A E in E; and also join each of the points B, B′ and C, D, E. Because A B is perpendicular to A C, A D, its prolongation A B′ is also perpendicular to these [illegible] (13, I), and B A C, B′ A C are right angles, and likewise B A D, B′ A D. But these angles have their sides respectively equal; wherefore the subtending lines B C, B′ C and also B D, B′ D are respectively equal (11, I). Again, because the triangles B C D, B′ C D have their sides respectively equal, the angles B C D, B′ C D are equal; and these angles having C E common, and the sides B C, B′ C equal, are subtended by the equal lines B E, B′ E; wherefore the triangle B E B′ is isosceles, and B B′ or A B is perpendicular to the right line A E.



5. *A right line* is said to be *perpendicular to a plane*, when it is perpendicular to every right line which it meets in that plane; and conversely *the plane* is said to be *perpendicular to the right line.*

Corollary 1. A right line is perpendicular to a plane, when it is perpendicular to two right lines which it meets in that plane (4).

Cor. 2. There cannot be two perpendiculars to a plane from the same point, and on the same side of the plane. For such lines would be perpendicular to the common section of that plane with the plane passing through them, which is impossible (cor. 13, 17, I).

THEOREM II.

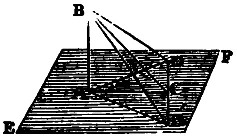
6. *If a right line be perpendicular to each of three right lines at a common point, those three right lines are in one and the same plane.*

Let A B be perpendicular to each of the lines A C, A D, A E meeting in the point A; these lines are in one and the same plane. (See the preceding figure).

For if not, A E one of these lines, and also the common section of the plane of A E, A B with the plane of the remaining lines A C, A D, would be perpendicular to A B, at the same point A, and in the same plane B A E; which is impossible (cor. 13, I).

THEOREM III.

7. *If a perpendicular to a plane meet either side of a right angle in the plane, the other side of this angle and an oblique line drawn from the perpendicular to its vertex, are perpendicular to each other.*



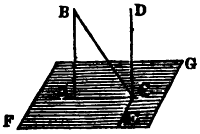
Let A B be a perpendicular to the plane E F, and A C one of the sides of the right angle A C D in this plane; the other side C D of this angle and the oblique line B C are respectively perpendicular.

Produce D C, and let C D′ be equal to C D. Join each of the points A, B and D, D′. Because the angles A C D, A C D′ are right angles and have equal sides, the subtending lines A D, A D′ are equal (11, I); and for a like reason, the lines B D, B D′ subtending the right angles B A D, B A D′ are also equal; wherefore the triangle D B D′ is isosceles, and B C, C D are respectively perpendicular (cor. 33, I).

Remark. The right lines A B, C D, though perpendicular to A C, are not parallel, as they would be were they situated in the same plane.

THEOREM IV.

8. *If two right lines be parallel, and one of them be perpendicular to a plane, the other is perpendicular to that plane.*



Let A B, C D be parallel, and A B perpendicular to the plane F G; C D is also perpendicular to this pl[illegible]

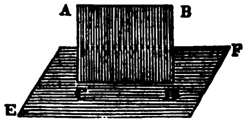
Join A C, B C; and in the plane F G, let the right line C E be perpendicular to A C. Because C E is perpendicular to the lines A C, B C (7), it is perpendicular to their plane (6); and because these lines are in the plane of the parallels A B, C D, the line C E is perpendicular to C D in this plane (4). But A C is perpendicular to A B and its parallel C D (22, I); wherefore C D being perpendicular to the lines C E, A C, is perpendicular to their plane F G (6).

Corollary 1. Two perpendiculars to the same plane are parallel. For otherwise, a right line drawn parallel to one of them from the point in which the other meets the plane, would be perpendicular to this plane; and there would be two perpendiculars to a plane at the same point, which is impossible (cor. 13, I).

Cor. 2. Two right lines which are parallel to a third right line, though not in the same plane with it, are also parallel. For a plane drawn perpendicular to this line will also be perpendicular to each of the other lines, which are therefore respectively parallel.

THEOREM V.

9. *If two right lines be parallel, and one of them be in a plane, the other cannot meet the plane.*



Let A B, C D be two parallels of which C D is in the plane E F; A B cannot meet this plane.

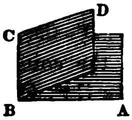
For if A B, which is in the plane A D passing through C D, could meet the plane E F, the point of meeting would be in C D, the common section of these planes. But A B cannot meet the line C D, since it is parallel to this line; wherefore A B cannot meet the plane E F.

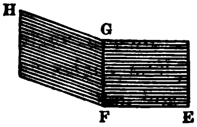
10. *A right line* is said to be *parallel to a plane*, when it cannot meet the plane; and conversely, *the plane* is said to be *paral*[illegible] *the right line*

§ 2. OF THE POSITIONS OF TWO DIVERGING PLANES.

11. When two planes meet each other, and are not in the same direction, they diverge and form an opening. The position of the planes relatively to each other depends on the opening formed; and this opening is evidently independent of the extension of the diverging planes.

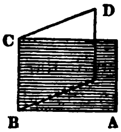
12. *A dihedral angle* is the opening formed by two diverging planes. The right line in which the planes meet, or their common section, is termed *the edge* of the dihedral angle formed; the diverging planes are called *sides* or *faces* of this angle.

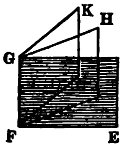




A dihedral angle is expressed by four letters, of which the second and third refer to its edge; and the first and fourth to its faces; as the dihedral angle A B C D or D B C A; the dihedral angle E F G H &c.

13. Dihedral angles are either equal or unequal. Two *equal* dihedral angles have their edges and faces respectively coincident, when placed on each other. Two *unequal* dihedral angles have two of their faces diverging, when their edges and their other faces are respectively coincident.

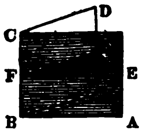


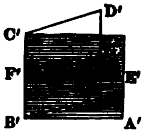


The dihedral angles A B C D, E F G H, of which the edges and faces coincide on superposition, are equal; the dihedral angles A B C D, E F G K are unequal, the latter being greater than F F G H by the dihedral angle H F G K.

14. Two equal or unequal dihedral angles may be determined by corresponding equal or unequal plane a[illegible]es, formed

by right lines drawn in their faces, perpendicularly to their edges. For when those plane angles are equal, the dihedral angles to which they correspond will evidently coincide and be equal; and when they have a given ratio to each other, the dihedral angles will have the same ratio, since planes passing through their edges will evidently divide them and the plane angles into the same number of equal corresponding angles.





Of the dihedral angles A B C D, A′B′C′D′, the corresponding plane angles are E F G, E′F′G′, the coincidence of which and that of the perpendicular edges C F, C′F′ evidently determine the coincidence of the faces A C, B D and A′C′, B′D′ respectively.

15. The *measure* of a *dihedral angle* is the *plane angle* formed by right lines drawn in its faces perpendicularly to its edge or the common section of its bounding planes. Thus the plane angle E F G is the measure of the dihedral angle A B C D; the plane angle E′F′G′ is the measure of A′B′C′D′, &c.

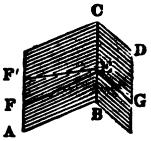
Remark. This definition being a legitimate inference from the equality and relations subsisting between dihedral angles and the corresponding plane angles formed by perpendiculars in their faces, it may be asserted that dihedral angles are right or oblique, and their faces perpendicular or oblique to each other, according as their corresponding plane angles are right or oblique; and also that adjacent and vertical dihedral angles possess properties analogous to those enunciated in Theorems I. and II. of the first Book.

THEOREM VI.

16. *If two planes form a dihedral right angle, a perpendicular in either plane to the common section, is perpen-*

dicular to the other plane; and conversely, a perpendicular to either plane, at any point of the common section, is in the other plane.

Let A C, B D be two planes forming the dihedral right angle A B C D, and let the right line E F in the plane A C be perpendicular to the common section B C; E F is perpendicular to the plane B D; and conversely.



In the plane B D, draw E G perpendicular to B C. Because the dihedral angle A B C D is a right angle, the corresponding plane angle F E G is also a right angle, and therefore E F is perpendicular to E G. But E F is likewise perpendicular to B C; wherefore it is perpendicular to the plane of E G, B C (cor. 5), and which is the plane B D.

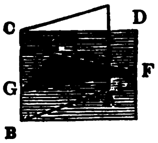
As to the converse, if E F considered as perpendicular to the plane B D, were not in the plane A C, a right line E F′ drawn perpendicular to B C, in the plane A C, would be perpendicular to the plane B D; and there would be two perpendiculars E F, E F′ to the plane B D, at the same point E, which is impossible (cor. 5).

Corollary 1. A plane which passes through a right line perpendicular to another plane, is perpendicular to this plane.

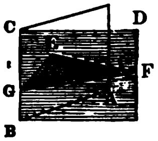
Cor. 2. The common section of two planes perpendicular to a third plane, is perpendicular to this plane.

THEOREM VII.

17. *If two planes form a dihedral oblique angle, and one of them be perpendicular to a right line, the other is oblique to this line.*



Let the planes A C, B D form the dihedral oblique angle A B C D, and let one of them A C be perpendicular to the right line E F; the other plane B D is oblique to E F.



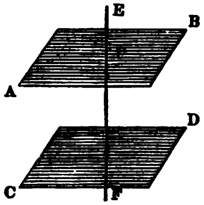
Join the points E, F in the planes A C, B D, and any point G in the common section B C. Because the plane A C is perpendicular to E F, the angle F E G is a right angle (5); and therefore the triangle F E G formed by the lines E F, F G, G E (1), has the angle E F G acute. But this angle is formed by E F and the right line F G in the plane B D; wherefore this plane is oblique to the right line E F.

3. OF PARALLEL PLANES.

18. It follows immediately from the preceding theorem, that two planes which are perpendicular to the same right line, cannot meet on either side of this line; for in such case, one of them meeting the other and forming an angle, would be oblique and not perpendicular to the right line.

19. Two planes are said to be *parallel*, when they cannot meet, however far produced.

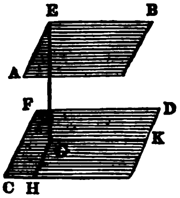
Corollary. Two planes are parallel, when they are perpendicular to the same right line.



The planes A B, C D regarded as incapable of meeting each other, or as perpendicular to the same right line E F, represent parallel planes.

THEOREM VIII.

20. *Two planes are parallel, when two diverging right lines in one of them are respectively parallel to two diverging lines in the other.*

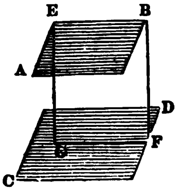


Let A B, C D be two planes, in which the diverging right lines E A, E B are respectively parallel to F C, F D; the planes A B, C D are parallel.

At the point E, let the right line E G be perpendicular to the plane C D, and through G in this plane draw the right lines G H, G K respectively parallel to F C, F D. Because E A, G H are parallel to F C, they are parallel to each other; and for a like reason, E B, G K are also parallel (cor. 8). But G H, G K are perpendicular to E G (5); their parallels E A, E B are therefore perpendicular to E G (8); and the planes A B, C D being perpendicular to the same line E G, are respectively parallel (cor. 19).

THEOREM IX.

21. *If two parallel planes be cut by a third plane, the corresponding intersections are parallel.*



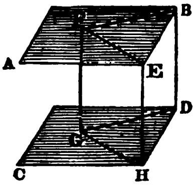
Let the two parallel planes A B, C D be cut by the plane

E F, and let E B, G F be the corresponding intersections; E B is parallel to G F.

For the right lines E B, G F in the same plane E F, cannot meet, though produced, since the parallel planes A B, C D, in which they are situated, do not meet. These lines or intersections E B, G F are therefore respectively parallel.

THEOREM X.

22. *If two parallel planes intersect the faces and edge of a dihedral angle, the corresponding plane angles formed by the intersections are respectively equal.*



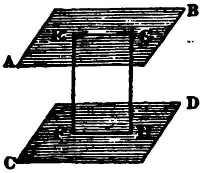
Let the two parallel planes A B, C D intersect the faces B G, E G and edge F G of the dihedral angle B F G H, and let the intersections be B F, D G and E F, H G; the angle B F E is equal to D G H.

Take F B, F E respectively equal to G D, G H and join B E, D H, and also B D, E H. Because the plane B G meets the parallel planes A B, C D, the sections B F, D G are parallel (21); but these lines are also equal; wherefore B D, F G joining their extremities are equal and parallel (27, I); and for a like reason, E H, F G are equal and parallel. Again, because B D, E H are parallel to F G, they are parallel to each other (cor. 8); but they are also equal; wherefore the connecting lines B E, D H are equal; and the two triangles F B E, G D H having their sides respectively equal, have the angle B F E equal to D G H (39, I).

THEOREM XI.

23. *Two parallel planes intercept equal parallel right lines.*

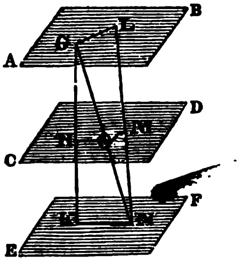
The parallel planes A B, C D intercept eqnal parallel right lines E F, G H.



Join E G and F H. Because E F, G H are parallel, they are in the same plane E H; and because this plane meets the parallel planes A B, C D, the intersections E G, F H are respectively parallel (21); wherefore E F G H is a parallelogram, and E F is equal to G H (77, I).

THEOREM XII.

24. *Three parallel planes intersecting two right lines, cut them proportionally.*



Let the planes A B, C D, E F cut the right lines G H K, L M N; these lines are cut proportionally, so that G H : H K :: L M : M N.

Join G L, K N in the planes A B, E F; draw G N meeting the plane C D in O, and join H O, M O in this plane. Because the plane G K N meets the parallel planes C D, E F, the sections H O, K N are parallel (21), and therefore G H : H K :: G O : O N (48, I). For a like reason, the sections O M, G L, formed by the plane N G L, and the parallel planes C D, A B, are also parallel; and therefore

G O : O N :: L M : M N. Compare this and the preceding proportion, and there will result G H : H K :: L M : M N.

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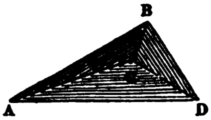
§ 4. OF THE POSITIONS OF THREE DIVERGING PLANES.

25. When three planes have a common point, but do not pass through the same right line, they evidently intersect each other, and form by their mutual divergence from that point, an opening which is called a *trihedral angle.* The point in which the diverging planes meet is the *vertex* of the trihedral angle formed; the planes forming this angle are its *sides* or *faces;* and the common sections of these faces are termed its *edges.*

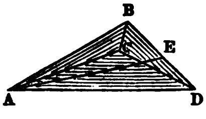


A trihedral angle is expressed by four letters, of which the first is placed at its vertex, and the three other letters are in the directions or at the extremities of its edges; as the trihedral angle A B C D, or A C D B, &c.

In a trihedral angle, there are evidently three *plane angles* formed by the divergence of its edges, and also three *dihedral angles* formed by the mutual inclinations of its diverging planes or faces. These angles are subjected to the conditions enunciated in the following theorems.

THEOREM XIII.

26. *In every trihedral angle, any two plane angles are together greater than the third.*

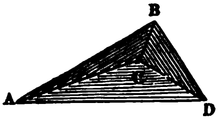


In the trihedral angle A B C D, any two plane angles B A C, C A D are together greater than the third plane angle B A D.

If either of the angles B A C, C A D be equal to B A D or greater than it, the theorem is evident. But if not, let the angle B A D be greater than either of the two angles B A C, C A D; and in the plane A B D, draw the right line A E, making the angle B A E equal to B A C. Draw also B D, meeting A E in E, and the edges A B, A D in B, D. Take A C equal to A E, and join B C, C D. Because the angles B A C, B A E are equal and have equal sides, B C is equal to B E (38, I); and because the sides B C, C D of the triangle B C D, are together greater than B D (35, I), the lines B E, C D are also greater than B D; and taking away B E, there remains C D greater than D E. Wherefore the triangles A C D, A E D have the angle C A D greater than E A D (cor. 43, I); and adding the angle B A C or B A E, the two angles B A C, C A D are together greater than B A E, E A D, or greater than the whole angle B A D.

THEOREM XIV.

27. *In every trihedral angle, the three plane angles are together less than four right angles.*

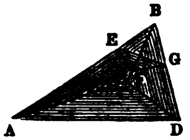


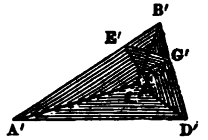
In the trihedral angle A B C D, the three plane angles B A C, B A D, C A D are together less than four right angles.

Let the faces of the trihedral angle A B C D be cut by a plane, and let B C D be the triangle formed by the lines of intersection. In the triangles A B C, A B D, A C D, at the vertex A, the nine plane angles are together equal to six right angles (33, I); but in each of the trihedral angles formed at the points B, C, D, two of those plane angles are together greater than the corresponding angle of the triangle B C D (26); wherefore the six plane angles at the bases of the triangles A B C, A B D, A C D, having the common vertex A, are together greater than the angles of the triangle B C D, which are equal to two right angles; and the remaining plane angles at this vertex are therefore less than four right angles.

THEOREM XV.

28. *If two trihedral angles have the three plane angles of the one respectively equal to the three plane angles of the other, the dihedral angles formed by the corresponding faces of the trihedral angles are respectively equal.*





Let A B C D, A′B′C′D′ be two trihedral angles, which have the plane angles B A C, B A D, C A D respectively equal to B′A′C′, B′A′D′, C′A′D′, and let C A B D, C′A′B′D′ be corresponding dihedral angles, of which the edges are A B, A′B′; the dihedral angle C A B D is equal to C′A′B′D′.

Take the several edges A B, A C, &c. of the trihedral angles A B C D, A′B′C′D′, all equal to one another, and join B, C, &c.. forming the isosceles triangles A B C, A B D, A C D respectively equal to A′B′C′, A′B′D′, A′C′D′ (38, I), and also the triangle B C D equal to B′C′D′ (41, I). In the edges A B, A′B′ of the dihedral angles C A B D, C′A′B′D′ take B E equal to B′E′, and through the points E, E′ let planes be drawn perpendicular to these edges and meeting B C, B′C′ and B D, B′D′, produced if necessary in the corresponding points F, F′ and G, G′. Join E, F, G and also E′, F′, G′. The right angled triangles B E F, B F G are respectively equal to B′E′F′, B′F′G′ (39, I), and therefore E F, E G are respectively equal to E′F′, E′G′. But the angles F B G, F′B′G′ of the triangles B C D, B′C′D′ are equal, and have equal sides B F, B′F′ and B G, B′G′; wherefore F G is equal to F′G′ (38, I), and the triangles E F G, E′F′G′ have the angle F E G equal to F′E′G′ (41, I). Now these equal angles measure the corresponding dihedral angles

C A B D, C′A′B′D′ (15); wherefore the dihedral angle C A B D is equal to the dihedral angle C′A′B′D′.

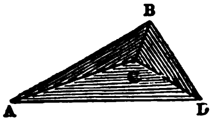
THEOREM XVI.

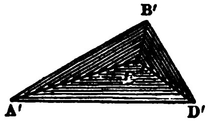
29. *If two trihedral angles have two plane angles, and the included dihedral angle of the one respectively equal to two similarly situated plane angles and the included dihedral angle of the other, these trihedral angles are in all respects equal.*

For the corresponding plane angles which contain the trihedral angles will evidently coincide, when the vertices and faces of the equal dihedral angles are respectively coincident, and the truth of the theorem is obvious.

THEOREM XVII.

30. *If two trihedral angles be contained by plane angles which are respectively equal and similarly situated, these trihedral angles are in all respects equal.*





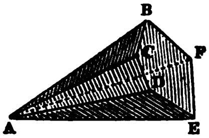
For the dihedral angles formed by equal and corresponding plane angles are respectively equal; and the trihedral angles will evidently coincide, when their vertices and two equal corresponding plane angles are coincident.

Remark. Trihedral angles which are contained by equal plane angles dissimilarly situated, are evidently incapable of coincidence by superposition, although their dihedral angles formed by equal corresponding plane angles, are respectively equal (28). These trihedral angles, relatively to those above considered, may be termed *reversed angles.* Their equality appears to be independent of the situation of their corresponding faces; it cannot be established by their coincidence, and must be regarded as a necessary consequence of the mutual equality of their constituent plane and dihedral angles.

Trihedral reversed angles were first noticed by Legendre, a mathematician of known distinction. In his geometry he calls them angles equal by symmetry, or simply symmetrical angles; but the term reversed here adopted has been preferred as more definite, and as more expressive of the characteristic difference between these angles and those having their bounding planes similarly posited. (See Lacroix's Geometry.)

§ 5. OF THE POSITIONS OF SEVERAL DIVERGING PLANES.

31. When four or a greater number of planes diverge from a common point and successively intersect each other, they form an opening, which is called *a polyhedral angle.* The point from which the planes diverge is the *vertex* of the polyhedral angle formed; the diverging planes are the *sides* or *faces* of this angle, and the common sections of these faces are its *edges.*



The polyhedral angle A B C D E F is contained by five planes or faces B A C, B A D, &c. diverging from the common vertex A, and forming by their successive intersections the edges A B, A C, &c.

32. The several plane angles or faces which constitute the boundaries of a polyhedral angle, form by their inclinations a like number of dihedral angles. These plane and dihedral angles may be considered in the same manner as those forming the subject of the preceding section; and by a like construction and mode of reasoning, the two following theorems may be easily demonstrated.

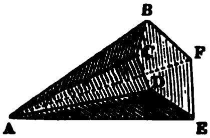
THEOREM XVIII.

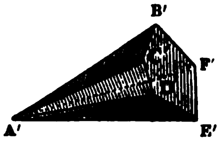
33. *The plane angles which contain a polyhedral angle are together less than four right angles.*

Remark. The polyhedral angles possessing the property here enunciated, are supposed to have all their dihedral angles salient, analogously to the plane angles noticed in article 72, Book I. (See the preceding figure.)

THEOREM XIX.

34. *If two polyhedral angles be contained by the same number of equal plane angles, similarly situated, and forming equal dihedral angles by their mutual inclinations, these polyhedral angles are respectively equal.*





Remark. A polyhedral angle is evidently determined by its plane angles, and the corresponding dihedral angles less three. The preceding theorem may therefore be more generally enunciated if considered necessary.

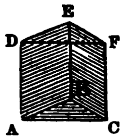
Should the polyhedral angles have their corresponding faces dissimilarly situated, they may be denominated *reversed* angles, analagously to the trihedral angles defined in the preceding section. Their equality viewed independently of the situations of their faces must be granted for the reasons there assigned.

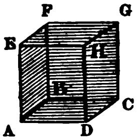
CHAPTER II.

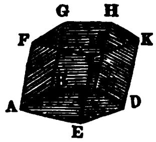
OF PRISMS.

35. WHEN three or more planes join the corresponding sides of two equal, similar, and parallel figures, they enclose, in connexion with those figures, a portion of space and constitute a solid, which is characterized by the positions and forms of its different boundaries.

36. A *prism* is a plane solid of which two of the bounding surfaces are equal, similar, and parallel rectilinear figures, and the remaining boundaries are parallelograms. The equal and parallel rectilinear figures are the *bases* of the prism; the connecting parallelograms are its *sides* or *faces*, and the intersections of consecutive faces are its *principal edges*.







The solid A B C E is a prism; the triangles A B C, D E F are its bases; the parallelograms A E, A F, B F are the faces or sides of the prism, and the parallel right lines A D, B E, C F are the principal edges.

The solids B D F H, B E G L are also prisms; the base of the first is the parallelogram A B C D; that of the second, the polygon A B C D E. The faces of both these prisms being parallelograms, their corresponding edges A E, B F, &c. are respectively parallel.

37. A prism takes the following denominations, of which the first two relate to the positions of its sides, and the last three to the figure of its base:

A *right prism* has its faces or sides respectively perpendicular to its base, as A B C D or B D F H; an *oblique prism* has its faces oblique to its base; as the prism B E G L.

Corollary. The faces of a right prism are evidently rectangles.

38. A *triangular prism* has its base a triangle, as the solid A B C E; a *quadrilateral prism* has its base a quadrilateral, as B D F H; and a *polygonal prism* has its base a polygon, as B E G L.

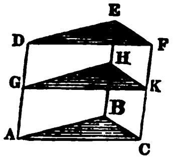
39. A quadrilateral or a polygonal prism may evidently be divided into triangular prisms, by diagonal planes, drawn through one of the edges; and conversely, two or more triangular prisms may compose a quadrilateral or polygonal prism. Triangular prisms may therefore be considered as

the component parts or elements of prisms viewed in general; and as such will be regarded as forming the leading objects of the present chapter.

§ 1. OF THE SECTION OF A TRIANGULAR PRISM BY A PLANE.

THEOREM I.

40. *If a triangular prism be cut by a plane parallel to the base, the section is a triangle equal in all respects to the base.*



Let A B C D E F be a triangular prism, and G H K a section parallel to its base; G H K is equal and similar to the base A B C.

Because the plane A E cuts the two parallel planes A B C, G H K, the intersection G H is parallel to A B. But the edge A D is parallel to B E; wherefore A B G H is a parallelogram, and G H is equal to A B. For a like reason, G K, H K are respectively equal and parallel to A C, B C; and the section G H K is therefore equal in all respects to A B C (41, I).

§ 2. OF THE SIDES AND ANGLES OF TWO TRIANGULAR PRISMS.

THEOREM II.

41. *If two triangular prisms have two planes, and the included angle of the one respectively equal to two similar and similarly situated planes and the included angle of the other, the remaining planes and angles are respectively equal.*

For the prisms will evidently coincide, when the equal dihedral angles and their diverging similar planes are respectively coincident.

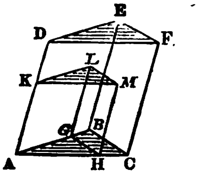
THEOREM III.

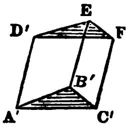
42. *If two triangular prisms have three planes containing an angle of the one respectively equal and similar to three similarly situated planes of the other, the remaining planes and angles are respectively equal.*

For the dihedral angles formed by the corresponding planes are equal (28); and the prisms will become identical by superposition.

THEOREM IV.

43. *If two triangular prisms have three planes containing an angle of the one, respectively similar to three similarly situated planes containing an angle of the other, the remaining planes are respectively similar, and their included angles respectively equal.*





Let A B C E, A′ B′ C′ E′ be two prisms in which the planes A B C, A E, A F are respectively similar to A′ B′ C′, A′ E′, A′ F′, the remaining planes are respectively similar, and their included angles respectively equal.

Take A G, A H, A K equal to A′B′, A′C′, A′D′, and complete the prism A G H K L M, which is equal to A′B′ C′E′ (41). The three planes A B C, A E, A F are therefore similar to A G H, A L, A M, and there results A B : A G :: B C : G H :: B E : G L; whence B C : B E :: G H : G L. But the planes B F, G M being respectively parallel, the angle C B E is equal to H G L (22), and it has been proved that the edges containing these angles are proportional; wherefore the parallelogram or face B F is similar to

G M, or to the face B′F′. Now the three plane angles at B are respectively equal to the plane angles at G; wherefore the trihedral angle B A D F is equal to G A K M or the trihedral angle B′A′D′F′ (30).

44. Two triangular prisms are said to be *similar*, when their bounding planes containing equal trihedral angles are respectively similar.

Corollary. The preceding theorem by virtue of this definition may be thus enunciated: *Two triangular prisms are similar, when their bounding planes containing equal trihedral angles are respectively similar.*

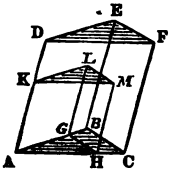
45. The *altitude* of a prism is a perpendicular drawn from one of its bases to the other base.

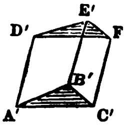
Corollary. The altitudes of two similar triangular prisms are proportional to any two corresponding principal edges. For the upper bases of the prisms being applied to each other, their lower bases will be parallel, and the perpendiculars and edges between these bases will evidently be proportional.

§ 3. OF THE AREAS OF TRIANGULAR PRISMS.

THEOREM V.

46. *The bases of similar triangular prisms are proportional to the squares of any two corresponding edges, or to the squares of the altitudes.*





Let A B C E, A′B′C′E′ be two similar prisms of which A D, A′D′ are corresponding edges; $ABC : A'B'C' :: AD^2 : A'D'^2$.

Take A G, A H, A K respectively equal to A′B′, A′C′, A′D′, and complete the prism A G H K L M, which is equal to A′B′C′ (41). The faces A E, A L being similar, $AB : AG :: AD : AK$, and therefore $AB^2 : AG^2 :: AD^2 : AK^2$ (the. 7). But the triangles A B C, A G H being also similar, $ABC : AGH :: AB^2 : AG^2$ (70, I); wherefore by equality of ratios, $ABC : AGH : AD^2 : AK^2$; and since A G H is equal to A′B′C′, and A K equal to A′D′, there results $ABC : A'B'C' :: AD^2 : A'D'^2$.

As to the altitudes of the prisms, they are proportional to the edges A D, A′D′ (cor. 45); and their squares are therefore proportional to the bases A B C, A′B′C′.

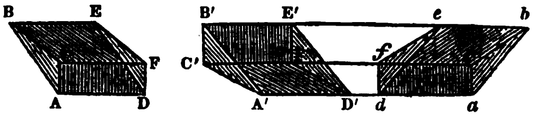
Remark. The proportion of the bases of the prisms is applicable to any other two corresponding boundaries, so that the areas of all these boundaries in each prism, are respectively proportional to the squares of two corresponding edges, or to the squares of the altitudes.

§ 4. OF THE EQUALITY OF TRIANGULAR PRISMS.

47. The conditions enunciated in articles 41, 42, evidently involve the equality of the spaces occupied by triangular prisms as well as that of their angles and bounding planes. But these conditions are not necessary when the equality of prisms is considered only as respects the spaces contained by their boundaries; for this equality may evidently subsist independently of the mutual coincidence of the planes and angles of the prisms.

THEOREM VI.

48. *Two triangular reversed prisms, which have equal bounding planes and plane angles, are respectively equal.*

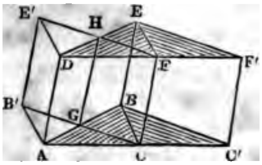


Let A B C F, A′B′C′F′ be reversed prisms, in which the corresponding edges C F, C′F′ are on opposite sides of the equal faces A B E D, A′B′E′D′, and of which the corresponding planes and plane angles are equal; the prism A B C F is equal to the prism A′ B′ C′ F′.

Produce the edges and faces of A′B′C′F′; take ad equal to A D; and making the angles bad, cad respectively equal to B A D, C A D, complete the prism $abcf$. Because the plane angles at A and A′ are respectively equal, the dihedral angle formed by the planes C D, B D is equal to the dihedral angle formed by C′ D′, B′ D′; and because these corresponding planes are also equal, and the last two equal to af, ae, the prism ae is equal to A E. But the solid A′ B′ C′f is equal to D′ E′ F′ c, since their planes are similarly situated, and their angles respectively equal; wherefore the prism $abcf$, or the remaining part of the solid A′ C′ B′ c is equal to the remaining prism A′ B′ C′ F′. Now the prism $abcf$ is equal to A B C F; wherefore the prism A B C F is equal to the prism A′ B′ C′ F′.

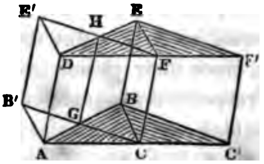
THEOREM VII.

49. *Two triangular prisms which have a common face, and equal contiguous bases in the same right line, are respectively equal.*



Let A B C E, C B C′E be triangular prisms having the common face B C E F, and the equal bases A B C, C B C′ in the same right line A C′; the prism A B C E is equal to the prism C B C′ E.

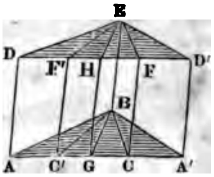
In the plane A B C′ produced, draw A B′ parallel and equal to B C; join B′ C by a right line meeting A B in G; and let A B′ C E′, A B′ G E′ be triangular prisms on the bases A B′C, A B′G, and therefore G B C E a triangular prism on the base G B C. Because the base A B′C and faces A E′, A F are respectively equal to the base C B C′ and faces C E, C F′, the prism A B′C E′ is equal to C B C′E; and because the base A B′G and faces A E′, A H are respectively equal to the base B C G or E F H and the faces E C, E G, the prism A B′G E′ is equal to the reversed prism G B C E. To each of these prisms add the adjacent prism A G C H, and the whole prism A B′C E′ is equal to A B C E. But it has been proved that the prism A B′C E′ is equal to the prism C B C′E; wherefore the prism A B C E is equal to C B C′E.



Corollary. A plane passing through any principal edge of a prism and bisecting the base, divides the prism into two equal parts.

THEOREM VIII.

50. *Two triangular prisms, which have a common principal edge and equal bases in the same right line, are respectively equal.*

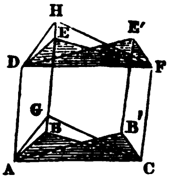


Let A B C E, A′B C′E be triangular prisms having the common edge B E, and the equal bases A B C, A′B C′ in the same right line A A′; the prism A B C E is equal to A′B C′E.

In C C′ let C G be equal to C′G; through G draw G H parallel to B E, and join B G, E H. Because A C is equal to A′C′ and C G equal to C′G, the remainders A G, A′G are equal, and the triangles or bases A B G, G B C are respectively equal to the bases A′B G, G B C′. But the prisms on these bases have the common face B G E H; wherefore the prism A B G E is equal to A'B G E, and the prism G B C E is equal to G B C'E (49). Add these prisms together, and the whole prism A B C E is equal to the corresponding prism A'B C'E. In like manner may the equality of A B C E and A'B C'E be proved, should the faces B E C F, B E C'F' be interchanged, by merely reading wholes for remainders and the contrary.

THEOREM IX.

51. *Two triangular prisms which have a common face, and the principal edges opposite to this face in a plane parallel to it, are respectively equal.*

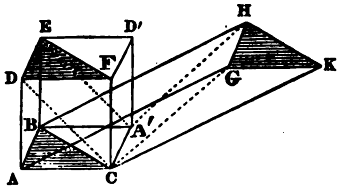


Let A B C E, A B'C E' be triangular prisms having the common face A C D F, and the edges B E, B'E' in a plane parallel to this face; the prism A B C E is equal to A B'C E'.

If the bases A B C, A B'C be in the same plane, they are equal; and the prism A B C E may evidently be applied to A B'C E', so that the edges may be coincident, and the equal bases in the same right line; in which case the prisms are equal (50). But if the bases A B C, A B'C be in different planes, produce either of them as A B'C and the corresponding base D E'F, until they meet the edge B E produced, in the points G, H. Join A G, G C and also D H, H F. The prism A G C H is, conformably to the preceding case, equal to A B'C E'; and the small solid A B C G is evidently equal in all respects to D E F H; wherefore the prism A G C H is equal to A B C E. But the prism A G C H is equal to A B'C E'; wherefore the prism A B C E is also equal to A B'C E'.

THEOREM X.

52. *Two triangular prisms on the same base and between the same parallel planes are respectively equal.*



Let A B C E, A B C H be triangular prisms on the same base A B C, and having their upper bases D E F, G H K in the same plane parallel to A B C; the prism A B C E is equal to A B C H.

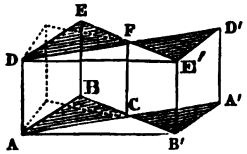
Complete the parallelograms A B A′C, D E D′F, and join C D, C G and A′E, A′H. Because the triangle C D A is equal to A′E B, or to the dissimilarly posited triangle E A′D′, and the faces C B, D B are respectively equal to the faces E F, A′F, the triangular prisms C D A E, E A′D′C are reversed and equal to each other (48). For the same reason, the reversed prisms A B C E, D′F E C are also equal. But the first two of these equal prisms compose the solid A A′D D′ and the last two also compose the same solid; wherefore these prisms are respectively equal, and accordingly the prism A B C E is equal to C D A E. In like manner it may be proved that the prism A B C H is equal to A G C H, which is equal to the prism C D A E, having the common face A B A′C parallel to the edges D E, G H, according to theorem IX.; wherefore the prism A B C E is equal to the prism A B C H.

Corollary. A triangular *oblique prism* is equal to a triangular *right prism* on the same base and between the same parallel planes, or having the same altitude.

THEOREM XI.

53. *Two triangular right prisms having equal bases and altitudes are respectively equal.*

Let A B C E, A'B'C E' be triangular right prisms having the equal bases A B C, A'B'C, and the same altitude; the prism A B C E is equal to A'B'C E'.



Place the prisms so that their faces A F, A'F may form the parallelogram A D'; let the edges B E, B'E' be on opposite sides of A D', and join A B', D E'. The angles A C B, A'C B' may be equal or unequal. In the first case, B C B' is a right line, and the bases A B C, A'B'C being equal, the triangle A B B' is equal to A A'B', and a right line joining B A' is parallel to A B'. The triangular prisms A B B'E, A A'B'E', having a common face A E' parallel to the plane of the edges B E, B'E', are therefore equal (48), and taking from each the common prism A C B'F, there remains the prism A B C E equal to A'B'C E'. In the case in which the angles A C B, A'C B' are unequal, a triangle may be formed on A C, so that one of its sides may be in the same right line with C B', and its vertex in a right line drawn through B parallel to A C. This triangle will be equal to A B C; and the corresponding prism having the face A F will be equal to A B C E (48), and also equal, by the first case, to the prism A'B'C E'; wherefore the prism A B C E is equal to A'B'C E'.

Corollary. A triangular *oblique prism* is equal to a triangular *right prism* having an equal base and the same altitude (cor. 52); and triangular prisms having equal bases and altitudes are respectively equal.

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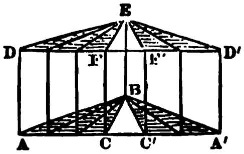
§ 5. OF THE PROPORTION OF THE VOLUMES OF TRIANGULAR PRISMS.

54. The possibility of dividing triangular prisms into equal parts follows from the preceding theorems and from those respecting the equality of triangles. These solids may

therefore be referred to measuring units, and their relations may be comprehended in the doctrine of algebraical proportion.

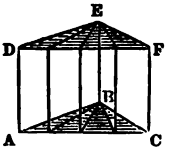


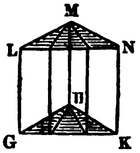
Of the preceding prisms A B C E, A'B C'E, the first contains three equal parts, and the second two. The ratio of these prisms is therefore as 3 : 2 ; whence the proportion A B C E : A'B C'E :: 3 : 2, in which the word prism is understood before A B C E and A'B C'E.

55. The *volume* of a prism is the number of times it contains a particular unit. The volume of the prism A B C E is 3, that of A'B C'E is 2; and the ratio of these volumes is the same as the ratio of the corresponding prisms.

THEOREM XII.

56. *Two triangular prisms which have equal altitudes are proportional to the areas of their bases.*



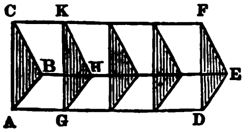


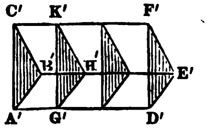
Let A B C E, G H K M be triangular prisms which have equal altitudes B E, H M; the prism A B C E : G H K M :: the area of A B C : the area of G H K.

For if the base A B C contains four units of which G H K contains three, so that A B C : G H K :: 4 : 3, and if planes be drawn through the several lines of division and the corresponding edges B E, H M, the prism A B C E will be divided into four equal parts, and the prism G H K M into three; wherefore A B C E : G H K M :: 4 : 3. But A B C : G H K :: 4 : 3; therefore A B C E : G H K M : A B C : G H K.

THEOREM XIII.

57. *Two triangular prisms are proportional to the products of their bases and altitudes.*



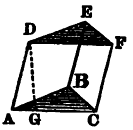


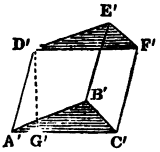
Let A B C E, A'B'C'E' be two triangular prisms, of which A D, A'D' are the altitudes; A B C D : A'B'C'D' :: A D.A B C : A'D'. A'B'C'.

For if A D contains four units and A'D' three, so that A D : A'D' :: 4 : 3; and if through the corresponding points of division, planes be drawn parallel to the bases A B C, A'B'C', the sections formed will be respectively equal to these bases (40); and the prism A B C E will be divided into four equal parts, and A'B'C'E' into three. Let A B C H be one of the equal parts of the prism A B C E, and A'B'C'H' a corresponding part of the prism A'B'C'E'. The altitudes A G, A'G' of these parts being equal, A B C H : A'B'C'H' :: A B C : A'B'C' (56), which multiplied by the proportion 4 : 3 :: A D : A'D' above given, there results 4 A B C D : 3 A'B'C'D' :: A D.A B C : A'D'. A'B'C' (th. 7). But the prism A B C E is equal to four times the prism A B C H, and the prism A'B'C'E' is three times A'B'C'H'; wherefore A B C E : A'B'C'E' :: A D.A B C : A'D'. A'B'C'.

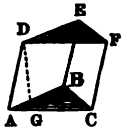
THEOREM XIV.

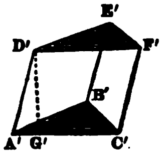
58. *Two similar triangular prisms are proportional to the cubes of their corresponding edges.*





Let A B C E, A'B'C'E' be similar triangular prisms, of which A D, A′D′ are corresponding edges, and D G, D′G′ the altitudes; $ABCE : A'B'C'E' :: AD^2 : A'D'^2$.



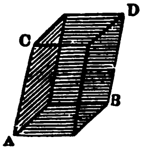


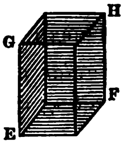
Because similar prisms have their bases proportional to the squares of their corresponding edges (46), $ABC : A'B'C' :: AD^2 : A'D'^2$, which multiplied by the proportion $DG : D'G' :: AD : A'D'$ (cor. 45), gives $ABC.DG : A'B'C'.D'G' :: AD^2 : A'D'^2$. But $ABCE : A'B'C'E' :: ABC.DG : A'B'C'.D'G'$ (57); wherefore $ABCE : A'B'C'E' :: AD^2 : A'D'^2$.

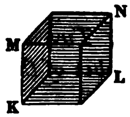
Corollary. Two similar triangular prisms are proportional to the cubes of the perimeters of their bases, or of any two corresponding boundaries, analogously to cor. 70, Book I.

§ 6. OF QUADRILATERAL PRISMS.

59. The several properties of quadrilateral prisms considered in this section are presented in a manner analogous to the properties of quadrilaterals treated in Chap. III. Book I. The particular solids to which they correspond are the following, the names of which refer to the positions and forms of their bounding surfaces:







60. A *parallelopiped* is a quadrilateral prism, of which the base is a parallelogram; as A B C D.

The opposite-faces of a parallelopiped are evidently equal and parallel, and a diagonal plane divides it into two equal reversed prisms.

Remark. This solid defined with relation to the figure of its base may be termed a *rhomboidal* or a *rhombic* prism, according as this base is a rhomboid or a rhombus (77, 78, I).

61. A *rectangular* parallelopiped has every two of its adjacent planes perpendicular to each other; as E F G H.

The bases of a rectangular parallelopiped are squares or rectangles, and its faces are rectangles.

Remark. This solid defined with respect to the figure of its bases, and the positions of its sides, may be termed a *right square* prism or a *right rectangular* prism, according as these bases are squares or rectangles.

62. A *cube* is a parallelopiped of which the bounding planes are all equal squares; as K L M N.

The angles of a cube are right angles, and this solid may be termed a regular *hexahedron.*

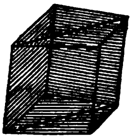
THEOREM XV.

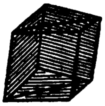
63. *Two quadrilateral prisms which are composed of triangular prisms having equal angles and faces similarly situated, have their angles and faces respectively equal.*

For the dihedral angles of the quadrilateral prisms are composed of equal dihedral angles contained by equal faces, and the solids evidently become identical when one is applied to the other. See article 82, Book I.

THEOREM XVI.

64. *Two quadrilateral prisms which are composed of similar triangular prisms similarly situated, are equiangular, and have their homologous edges proportional.*

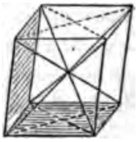




For the bases of the quadrilateral prisms being composed of similar triangles are equiangular (83, I), and these bases being therefore similar, and also the faces of the prisms at corresponding edges, the plane and trihedral angles are respectively equal, and the homologous edges are proportional (43).

THEOREM XVII.

65. *The squares of the four diagonals of a parallelopiped are together equal to the squares of the twelve edges.*



For the squares of two diagonals of the parallelopiped in the same diagonal plane, are together equal to the squares of two principal edges and two diagonals of the bases (82, I); wherefore the squares of the four diagonals of the parallelopiped are together equal to the squares of the four principal edges and the four diagonals of the bases. But the squares of these latter diagonals are together equal to the squares of the eight corresponding edges of the bases; wherefore the squares of the four diagonals of the parallelopiped are together equal to the squares of its twelve edges.

THEOREM XVIII.

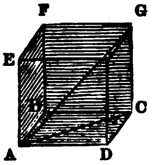
66. *Two parallelopipeds which have equal bases and altitudes are respectively equal.*

For diagonal planes divide the parallelopipeds into triangular prisms which have equal bases and altitudes, and which are respectively equal (cor. 53). The parallelopipeds being composed of these equal triangular prisms are therefore respectively equal.

Corollary. A parallelopiped is equal to a rectangular parallelopiped having the same base and the same altitude; and a parallelopiped or a rectangular parallelopiped is equal to twice a triangular prism on half the base and between the same parallel planes.

THEOREM XIX.

67. *The square on any of the diagonals of a rectangular parallelopiped is equal to the three squares on the adjacent edges.*



Let A C E G be a rectangular parallelopiped, of which A G is a diagonal; the square on A G is equal to the three squares on the adjacent edges A B, A D, A E.

Draw the diagonal A C of the base A B C D. Because C G is perpendicular to the base of the solid, A C G is a right angled triangle, and therefore the square on A G is equal to the two squares on A C, C G (87, I); and because A B C is also a right angled triangle, the square on A C is equal to the two squares on A B, B C; wherefore the square on A G is equal to the squares on A B, B C, C G. But B C, C G are respectively equal to A D, A E; wherefore the square on the diagonal A G is equal to the squares on the adjacent edges A B, A D, A E.

THEOREM XX.

68. *Two similar quadrilateral prisms are proportional to the cubes of their homologous edges.*

For these solids are proportional to their similar component triangular prisms, formed by corresponding diagonal planes, and are therefore proportional to the cubes of their homologous edges (58).

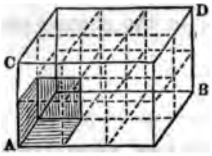
Corollary. Two similar quadrilateral prisms are proportional to the cubes of the perimeters of their several bounding planes; for these perimeters are evidently proportional to any two of the corresponding edges of which they are composed.

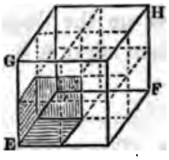
THEOREM XXI.

69. *Two rectangular parallelopipeds are proportional to the products of their bases and altitudes.*

For diagonal planes divide those solids into triangular prisms which are respectively proportional to the products of their corresponding bases and altitudes (57). The rectangular parallelopipeds composed of these triangular prisms are therefore proportional to like products of their bases and altitudes.

Remark. By virtue of this theorem, the contents or volumes of quadrilateral prisms may be determined conformably to the established mode of measuring solids. Assume for the unit of volumes, a cube of which the side of the base is the linear unit of length, and therefore the area of the base, the corresponding unit of surfaces. The volume of this cube being 1, and the product of its altitude and the area of its base being also 1, the corresponding volume of a rectangular parallelopiped in terms of cubic units may be determined by the product of the numbers which express its base and altitude, and may be represented by this product.





Thus the volume of the rectangular parallelopiped A B C D, of which the base A B contains 6 square units, and the altitude A C contains two linear units, may be represented by the product $6.2=12$, which evidently expresses the number of cubic units contained in the solid A B C D.

In like manner, the volume of the cube E F G H, relatively to the assumed unit, becomes 4.2 or $2^3=8$; and this solid may therefore be represented by the third power or cube of the side of its base E F.

The volume of a triangular prism immediately results from that of the corresponding rectangular parallelopiped between the same parallel planes. It is evidently equal to the product of the altitude of the prism and the area of its base; and by composition, the volumes of quadrilateral prisms may be expressed in terms of cubic units by the products of their altitudes and the areas of their bases.

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§ 7. OF POLYGONAL PRISMS.

70. The particular forms of polygonal prisms are distinguished by denominations relative to their bases, and the general properties of these solids may be presented in a manner analogous to the properties of polygons considered in Chap. IV., Book I.

THEOREM XXII.

71. *Two polygonal prisms which are composed of the same number of triangular prisms, having equal angles and planes similarly situated, have their angles and bounding planes respectively equal.*

For the corresponding dihedral angles of the polygonal prisms are composed of equal dihedral angles having equal and similar faces, and these solids become identical when their equal and similar bases are coincident. See article 96, Book I.

THEOREM XXIII.

72. *Two polygonal prisms which are composed of similar triangular prisms similarly situated, are respectively equiangular, and have their homologous edges proportional.*

The demonstration of this theorem is similar to that of (64), which may be easily extended to any number of faces, by comparing in succession the corresponding triangular prisms which compose the polygonal prisms.

THEOREM XXIV.

73. *Two similar polygonal prisms are proportional to the cubes of their homologous edges.*

For their component similar triangular prisms are proportional to the cubes of the homologous edges, which form the corresponding boundaries of the bases and sides of the polygonal prisms.

THEOREM XXV.

74. *The volume of a polygonal prism, referred to cubic units, is equal to the product of its altitude and the area of its base.*

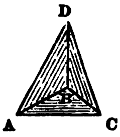
For the volumes of its component triangular prisms are respectively equal to the product of its altitude and the areas of their bases, which together compose the base of t[illegible]ygonal prism.

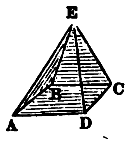
CHAPTER III.

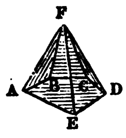
OF PYRAMIDS.

75. When three or more planes meeting in a point, join the sides of a rectilinear figure, they enclose, in connexion with this figure, a portion of space, and constitute a solid, which is characterized by the positions and forms of its different boundaries.

76. A *pyramid* is a plane solid, of which one of the bounding surfaces is a rectilinear figure, and the remaining boundaries are triangles meeting in a common point. The rectilinear figure is the *base* of the pyramid; the common point is its *vertex;* the connecting triangles are its *sides* or *faces;* and the intersections of consecutive faces are its *principal edges.*







The solid A B C D is a pyramid; the triangle A B C is its base; the point D its vertex; the triangles A D B, A D C, B D C are the faces of the pyramid, and the right lines A D, B D, C D are the corresponding principal edges.

The solids B D E, B E F are also pyramids; the base of the first is the parallelogram A B C D; that of the second, the polygon A B C D E.

77. A pyramid takes the following denominations relative to the figure of its base:

A *triangular* pyramid has its base a triangle; as the solid A B C D: this pyramid is frequently called a *tetrahedron;* a *quadrilateral* pyramid has its base a quadrilateral, as B D E; and a *polygonal* pyramid has its base a polygon, as the solid B E F.

78. A quadrilateral or a polygonal pyramid may evidently be divided into triangular pyramids by diagonal planes drawn through one of its edges; and conversely, two or more triangular pyramids may compose a quadrilateral or a polygonal pyramid. Triangular pyramids may therefore be considered as the component parts or elements of prisms viewed in general, and as such will form the leading objects of the present chapter.

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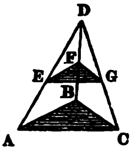
§ 1. OF THE SECTIONS OF TRIANGULAR PYRAMIDS BY PLANES.

THEOREM I.

79. *If a triangular pyramid be cut by a plane parallel to its base, the section is a figure similar to the base.*



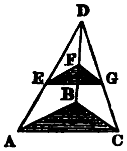
Let A B C D be a triangular pyramid, and E F G a section parallel to the base A B C; this section is similar to A B C.

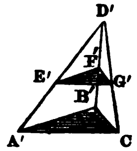
Because the parallel planes E F G, A B C intersect the faces B A D, C A D and the edge A D of the dihedral angle

B A D C, the angle F E G is equal to B A C (22); and for a like reason, the angle E G F is equal to A C B; wherefore the triangles E F G, A B C are equiangular and similar (31, 50, I).

THEOREM II.

80. *If two triangular pyramids on equal bases and between the same parallel planes, be cut by a plane parallel to the base, the sections are respectively equal.*





Let A B C D, A′B′C′D′ be triangular pyramids on equal bases and between the same parallel planes; the sections E F G, E′F′G′ in a plane parallel to these bases, are respectively equal.

Because the triangles A B C, E F G are by the preceding theorem similar, A B C : E F G :: A B^2 : E F^2 (70, I); and because A B, E F are parallel, they are also proportional to A D, E D; wherefore A B C : E F G :: A D^2 : E D^2, and for a like reason A′B′C′ : E′F′G :: A′D′^{2} : E′D′^{2}. But the edges A D, A′D′ being cut proportionally, A D : A′D′ :: E D : E′D′ (24); wherefore by equality of ratios, A B C : A′B′C′ :: E F G : E′F′G′, and since A B C is equal to A′B′C′, it follows that E F G is equal to E′F′G′.

Remark. The last proportion is independent of the equality of the bases A B C, A′B′C′, and will therefore apply to any two triangular pyramids between the same parallel planes.

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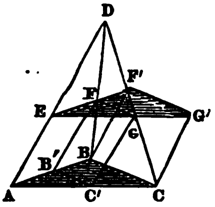
§ 2. OF TRIANGULAR PRISMS DESCRIBED IN AND ABOUT PYRAMIDS.

81. When the upper base of a prism is a section of a pyramid parallel to its base, and one of the principal edges of the prism forms a part of a principal edge of the pyramid, the *prism* is said to be *inscribed in the pyramid;* and when

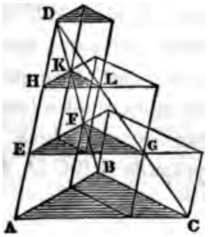
the lower base of a prism is the base of a pyramid or a section parallel to this base, and one of the principal edges of the prism forms a part of a principal edge of the pyramid, the *prism* is said to be *described about the pyramid.*



Of the preceding solids, A B'C'F is a triangular prism inscribed in the triangular pyramid A B C D, and A B C F' is a triangular prism described about, or circumscribing, the same pyramid.

THEOREM III.

82. *If a series of triangular prisms of the same altitude be inscribed in a triangular pyramid, and a corresponding series be described about the pyramid, the difference of these series is equal to the lowest circumscribed prism on the base of the pyramid.*

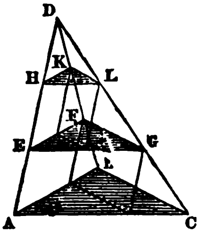


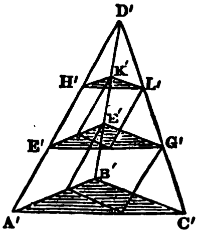
For the upper base E F G of any inscribed prism is the lower base of a circumscribing prism of equal altitude; and each inscribed prism taken in order from the base of the pyramid to its vertex, is therefore equal to the circumscribed prism immediately above it (cor. 53). But the number of inscribed prisms is equal to that of the circumscribed prisms, with the exception of the prism on the base of the pyramid; wherefore the difference between the series of inscribed and circumscribed prisms is equal to the lowest circumscribed prism on the base of the pyramid.

Corollary. The difference between the series of inscribed prisms and the pyramid is evidently less than the lowest circumscribed prism; and this series will continually approach the pyramid by continually diminishing their equal assumed altitudes.

THEOREM IV.

83. *If two series of triangular prisms of the same altitude be inscribed in triangular pyramids on equal bases and between the same parallel planes, the several corresponding prisms in these series are respectively equal.*





For the upper bases E F G, E'F'G' of any two corresponding inscribed prisms are equal sections of the pyramids A B C D, A'B'C'D' (80), and these prisms are equal, since they have equal bases and altitudes (cor. 53); wherefore the several corresponding prisms which constitute the series inscribed in the pyramids A B C D, A'B'C'D'. are respectively equal.

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§ 3. OF THE SIDES AND ANGLES OF TWO TRIANGULAR PYRAMIDS.

THEOREM V.

84. *If two triangular pyramids have two planes and the included angle of the one respectively equal to two similar*

and similarly situated planes and the included angle of the other, the remaining planes and angles are respectively equal.

For the pyramids will evidently coincide, when the equal dihedral angles and the diverging similar planes are respectively coincident.

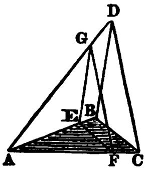
THEOREM VI.

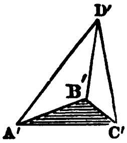
85. *If two triangular pyramids have three planes containing an angle of the one, respectively equal and similar to three similarly situated planes in the other, the remaining planes and also the angles are respectively equal.*

For the dihedral angles formed by the corresponding planes are equal (28), and the pyramids will become identical when either is applied to the other.

THEOREM VII.

86. *If two triangular pyramids have three planes containing an angle of the one, respectively similar to three similarly situated planes containing an angle of the other, the remaining planes are similar, and the angles are respectively equal.*





Let A B C D, A'B'C'D' be triangular pyramids, in which the planes A B C, A D C, B D C are respectively similar to A'B'C', A'D'C', B'D'C'; the remaining planes are similar, and the angles are respectively equal.

Take A E, A F, A G equal to A'B', A'C', A'D', and complete the pyramid A E F G, which is equal to A'B'C'D' (85). The three planes A B C, A B D, A C D are therefore similar to A E F, A E G, A F G, and there results A B : A E :: B C : E F :: B D : E G; whence B C : B D :: E F : E G. But the planes B C D, E F G being respectively parallel, the angle C B D is equal to F E G (22), and it has been proved that the edges containing these angles are proportional, wherefore the triangle or face C B D is similar to F E G or to C′B′D′. Now the three plane angles at B are respectively equal to the plane angles at E; wherefore the trihedral angle B A C D is equal to E A F G or B′A′C′D′ (30).

87. Two triangular pyramids are said to be *similar*, when their bounding planes containing equal trihedral angles, are respectively similar.

Corollary. The preceding theorem, by virtue of this definition, may be thus enunciated: *Two triangular pyramids are similar, when their bounding planes containing equal trihedral angles, are respectively similar.*

88. The *altitude* of a pyramid is a perpendicular drawn from its vertex to the base.

Corollary. The altitudes of two similar triangular pyramids are proportional to any two corresponding principal edges. For the vertices of the pyramids being applied to each other and also any two corresponding planes, the bases will be parallel, and the perpendiculars and edges between these bases will be evidently proportional.

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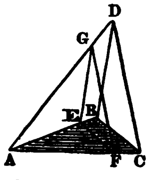
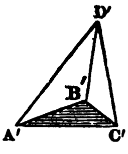
§ 4. OF THE AREAS OF TRIANGULAR PYRAMIDS.

THEOREM VIII.

89. *The bases of similar triangular pyramids are proportional to the squares of any two corresponding edges, or to the squares of the altitudes.*





Let A B C D, A′B′C′D′ be similar triangular pyramids, of which A D, A'D' are corresponding edges, A B C : A'B'C' :: A D^2 : $A'D'^2$.

Take A E, A F, A G respectively equal to A'B', A'C', A'D', and complete the pyramid A E F G, which is equal to A'B'C'D' (85). The faces A D B, A G E being similar, A B, A E are proportional to A D, A G, and therefore A B^2 : A E^2 :: A D^2 : A G^2 (th. 7). But the triangles A B C, A E F being also similar, A B C : A E F :: A B^2 : A E^2 (70, I); wherefore by equality of ratios, A B C : A E F :: A D^2 : A G^2; and since A E F is equal to A'B'C' and A G equal to A'D', there results A B C : A'B'C' :: A D^2 : $A'D'^2$.

As to the altitudes of the pyramids, they are proportional to the principal edges A D, A'D'; and their squares are therefore proportional to the bases A B C, A'B'C'.

Remark. The proportion of the bases is applicable to any two corresponding sides of the pyramid, so that the areas of all the boundaries in each pyramid are respectively proportional to the squares of two corresponding edges, or to the squares of the altitudes.

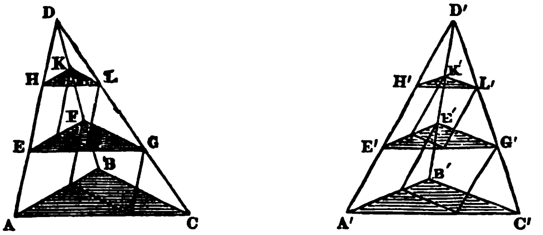
§ 5. OF THE EQUALITY OF TRIANGULAR PYRAMIDS.

90. The conditions enunciated in Theorems v. vi., evidently involve the equality of the spaces occupied by triangular pyramids, as well as that of their angles and bounding planes. But these conditions are not necessary when the equality of pyramids is considered only as respects the spaces contained by their boundaries; for this equality may subsist independently of the mutual coincidence of the several planes and angles of the pyramids.

THEOREM IX.

91. *Two triangular pyramids which have equal bases and altitudes, are respectively equal.*

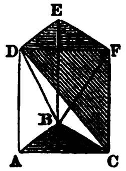
Let A B C D, A'B'C'D' be triangular pyramids, which have equal bases A B C, A'B'C', and equal altitudes terminating in the vertices D, D′; the pyramid A B C D is equal to the pyramid A′B′C′D′.



For if two series of triangular prisms of the same altitude be inscribed in the pyramids A B C D, A′B′C′D′, they will be respectively equal (83), whatever be the number of their component prisms, and however near they approach the pyramids A B C D, A′B′C′D′, which form their limits (cor. 82), and between which there must subsist a corresponding equality. Wherefore the pyramid A B C D is equal to the pyramid A′B′C′D′.

THEOREM X.

92. *Every triangular pyramid is equal to the third part of a prism, having the same base and altitude.*



Let A B C D be a triangular pyramid, and A B C E a corresponding prism having the same base and altitude; the pyramid A B C D is equal to a third part of the prism A B C E.

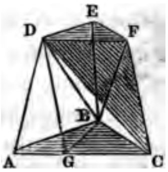
Join the vertex of the base A B C and the point F. Because the upper base D E F of the prism is parallel and equal to A B C, the pyramid D E F B is equal to A B C D (91); and because the triangle C D F is equal to C D A, the pyramid C D F B is equal to C D A B. But the pyramid C D A B is the same as the pyramid A B C D; wherefore the pyramid

C D F B is equal to A B C D, and the three pyramids A B C D, D E F B, C D F B are respectively equal. Now these pyramids evidently compose the prism A B C E; wherefore any one of them, as A B C D, is equal to a third part of the prism A B C E.

93. The *frustum* of a pyramid is the lower part of this solid, cut by a plane parallel to the base.

THEOREM XI.

94. *The frustum of a triangular pyramid is equal to three triangular pyramids, of which the altitude is the same as that of the frustum, and the bases are the upper and lower bases of this solid, and a mean proportional between these bases.*



Let A B C E be the frustum of a triangular pyramid, and let the triangle B G C, having its side G C a part of A C, be a mean proportional between the bases A B C, D E F; the frustum A B C E is equal to the triangular pyramids A B C D, D E F B, G B C D.

Join B D, B F, B G. Because the triangles A B C, G B C have the same vertex, they are proportional to A C, G C (66, I), and therefore A B C^2 : G B C^2 :: A C^2 : G C^2 (th. 7); and because G B C^2 is by hypothesis equal to A B C.D E F, this proportion becomes A B C : D E F :: A C^2 : G C^2. But the similar bases A B C, D E F are also proportional to A C^2, D F^2 (70, I); wherefore G C is equal to D F; and G D F C is a parallelogram in which the triangle G D C is equal to C D F. Now the frustum A B C E is composed of the pyramids A B C D, D E F B, C D F B, the last of which is equal to the pyramid C D G B or G B C D; wherefore the frustum A B C E is equal to the three pyramids A B C D, D E F B, G B C D.

§ 6. OF THE PROPORTION OF THE VOLUMES OF TRIANGULAR PYRAMIDS.

95. The comparison of triangular pyramids may in accordance with Theorem x. be directly instituted, by referring them to corresponding prisms. The following theorems will therefore be sufficiently obvious without the necessity of particular exemplification.

THEOREM XII.

96. *Two triangular pyramids which have equal altitudes are proportional to the areas of their bases.*

The demonstration of this follows from articles 92 and 56.

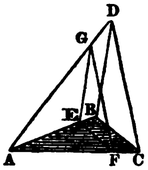
THEOREM XIII.

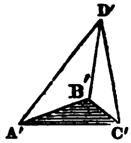
97. *Two triangular pyramids are proportional to the products of their bases and altitudes.*

For the pyramids are evidently proportional to their corresponding prisms having the same bases and altitudes (92); and are therefore proportional to the products of these altitudes and the areas of the bases (57).

THEOREM XIV.

98. *Two similar triangular pyramids are proportional to the cubes of their homologous edges.*





For the pyramids are proportional to corresponding similar prisms having the same bases and coincident homologous edges (92); and these prisms being proportional to the cubes of their homologous edges (58), the theorem is evident.

THEOREM XV.

99. *The volume of a triangular pyramid is equal to one third of the product of its altitude and the area of its base.*

For the volume of the pyramid is one third that of the corresponding prism, which is equal to the product of its altitude and the area of its base (74).

---

§ 7. OF QUADRILATERAL PYRAMIDS.

100. The particular denominations by which quadrilateral prisms are distinguished correspond to the different forms of their bases; and regarding these solids as composed of triangular pyramids, their chief properties may be presented without formal demonstration.

THEOREM XVI.

101. *Two quadrilateral pyramids which are composed of similar triangular pyramids similarly situated, are respectively equal.*

THEOREM XVII.

102. *Two quadrilateral pyramids which have equal bases and altitudes, are respectively equal.*

For the volumes of quadrilateral prisms having the same bases and altitudes are equal (66); and these prisms being each of them triple the triangular pyramids which compose the quadrilateral pyramids (92), the volumes of these solids are respectively equal.

THEOREM XVIII.

103. *Two quadrilateral pyramids are proportional to the products of their bases and altitudes.*

THEOREM XIX.

104. *Two similar quadrilateral pyramids are proportional to the cubes of their homologous edges.*

For these solids may evidently be divided into triangular pyramids, which are respectively similar and proportional to the cubes of the homologous edges of the quadrilateral pyramids.

THEOREM XX.

105. *The volume of a quadrilateral pyramid is equal to one third of the product of its altitude and the area of its base.*

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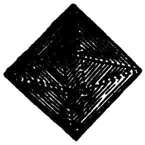
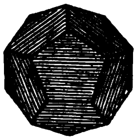
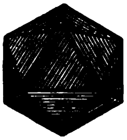
§ 8. OF POLYGONAL PYRAMIDS.

106. The different forms of polygonal pyramids are distinguished by denominations corresponding to their bases: their chief properties are precisely analogous to those of quadrilateral pyramids, and may if necessary be similarly enunciated.

Remark. Besides the plane solids considered in this and the preceding chapter, there are a variety of others which may be comprehended under the general denomination *polyhedron.* These solids are characterized by no particular positions of their boundaries; they are viewed without reference to bases, and their surfaces being any rectilinear figures whatever, the plane angles at their several vertices may constitute any number of trihedral and polyhedral angles. As to these polyhedrons treated under general forms, their properties are similar to those already noticed, and depend on their composition from pyramids at corresponding vertices. The *regular* forms which they assume are limited to three. These are determined by theorems XIV. XVIII.; from which it follows that a trihedral angle of any solid contained by equal planes, can be formed by means only of three equilateral triangles, or squares, or pentagons; and that a polyhedral angle can be similarly formed by means only of four or five equilateral triangles. The trihedral angles formed of equilateral triangles correspond to the *regular* triangular pyramid: those formed of squares correspond to the *regular* prism or cube. If these two solids be excluded from the class under consideration, there will remain *three regular* polyhedrons, the first bounded by eight equilateral triangles, forming tetrahedral angles; the second contained by twelve regular pentagons, constituting trihedral angles; and the third bounded by twenty equilateral triangles, forming pentahedral angles. The particular names of these solids are *octahedron, dodecahedron, icosahedron.* Their graphic representations are the following.







CHAPTER IV.

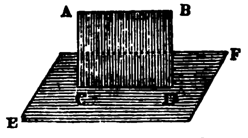
OF CONSTRUCTIONS AND MENSURATION RELATIVE TO PLANES AND PLANE SOLIDS.

107. THE two classes of problems here resolved exemplify the theoretical principles involved in the preceding chapters: the first class may be viewed as forming the basis of descriptive and practical solid geometry; the second may be regarded as constituting an important branch of the mensuration of solids.

§ 1. OF CONSTRUCTIONS RELATIVE TO PLANES.

PROBLEM I.

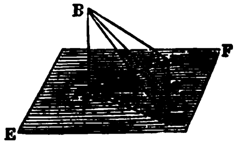
108. *Through a given point* A *to draw a right line parallel to a given plane* E F.



In the plane EF, draw any right line C D, and through the point A, draw in the plane of A and C D, the right line AB parallel to C D (103, I); this right line AB will be parallel to the given plane E F (9).

PROBLEM II.

109. *From a given point* A *or* B *to draw a perpendicular to a given plane* E F.



First, let the given point A be in the plane E F, and through it draw in this plane the right line A C. From C draw in E F the right line C D perpendicular to A C, and from the same point draw in any plane passing through C D, the right line C B perpendicular to C D. In the plane of A C, B C and perpendicularly to A C, draw the right line A B, which will be perpendicular to the given plane E F (7).

Secondly, let the given point B be without the plane E F, and draw in this plane any right line D D′. From B to D D′ draw the perpendicular B C; and in the plane E F draw C A perpendicular to C D. From B to C A draw the perpendicular B A, which will also be perpendicular to the plane E F (7).

Remark. The point B relatively to the plane E F is evidently determined by the perpendicular B A and the position of its extremity, or the point A. This point is termed the *orthogonal projection*, or simply the *projection* of the point B; and the plane E F is named the *plane of projection.*

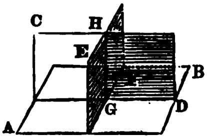
PROBLEM III.

110. *From the point* C *in which a given right line* B C *meets a given plane* E F, *to draw in this plane a perpendicular to the given line.*

Let the point A in the plane E F be the projection of B (rem. 109); join A C in this plane, and from C draw the right line C D perpendicular to A C. This line C D is evidently perpendicular to B C (7).

PROBLEM IV.

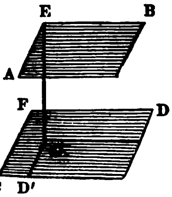
111. *Through a given point* D *or* G *to draw a plane perpendicular to a given right line* G H.



Through the given right line G H draw two planes C D, E F, the first passing through the given point, and forming with the second, any dihedral angle D G H F. In these planes draw the right lines D G, G F, each perpendicular to G H; the plane A B passing through these lines is perpendicular to the given line G H (4, 5).

PROBLEM V.

112. *Through a given point* E *to draw a plane parallel to a given plane* C D.



In the given plane C D draw any two right lines F C, F D meeting in F, and through the given point E draw the right lines E A, F B respectively parallel to F C, F D (103, I); the plane A B passing through these right lines is parallel to the given plane C D (20).

PROBLEM VI.

113. *Through a given point* A *to draw a plane perpendicular to a given plane* E F.

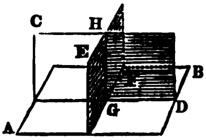
From the given point A to the plane E F draw the perpendicular A C (2), through which let any plane A D be drawn;

the plane AD is perpendicular to the given plane EF (cor. I, 16). See fig. prob. I.

Remark. The plane AD will evidently be perpendicular to two parallel planes, of which one is the plane E F.

PROBLEM VII.

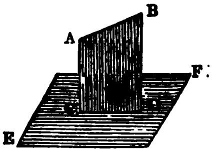
114. *Through a given point* B *to draw a plane perpendicular to two given planes* C D, E F.



From the given point B to the planes C D, E F draw the perpendiculars BD, BF (109). The plane AB passing through BD, BF is perpendicular to the given planes C D, E F.

PROBLEM VIII.

115. *Through a given right line* AB *to draw a plane perpendicular to a given plane* E F.

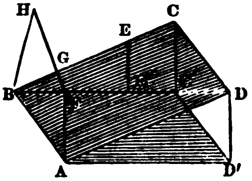


From the point A in the given right line AB draw AC perpendicular to the plane E F (109). The plane AB passing through AB, AC will be perpendicular to the given plane E F (cor. I, 16).

Remark. The right line C D joining the projections of the points A, B is in the perpendicular plane A D, and passes through the projection of every point in A B. It is therefore called the *projection* of AB; and the plane AD by which it may be determined is called the *projecting* plane. When the perpendiculars A C, B D are equal, A B is parallel to its projection C D; and when they are unequal, it is inclined to this projection and to the plane E F. The inclination of A B to C D is determined by the angles which they form when produced to meet each other; and this angle also determines the inclination of A B to the plane E F. It is evidently equal to the angle formed by A B, and a parallel to C D at the point A; and in conjunction with C D and the perpendicular A C, will determine both the position and length of A B.

PROBLEM IX.

116. *To find the common section of two planes, of which one is determined relatively to the other by the projections of three of its points, and by the corresponding perpendiculars.*



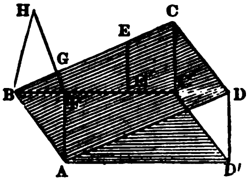
Let A C, A C′ be the planes; C′, D′, E′ the given projections, and C C′, D D′, E E′ the corresponding perpendiculars. Join C E, C′E′ which produce to meet in B; through D, D′ draw D A, D′A respectively parallel to C B, C′B; and join A B which will be the common section required.

For the parallels C B, D A are in the plane A C, and their projections C′B, D′A are in the plane A C′; wherefore the points B, A are in both the planes A C, A C′, and A B is the corresponding section.

PROBLEM X.

117. *To find the angle formed by the common section of two planes and a right line, which is determined in one of the planes by its projection in the other, and by two corresponding perpendiculars at its extremities.*

Let A B be the common section of the planes; C′E′ the projection of the right line, and C C′, E E′ the perpendiculars. Join C E, and let it and C′E′ produced meet the common section in B. From any point A of this section, draw A F perpendicular to B C′, and from F draw F G H perpendicular to B C. Make B H equal to B A, and G B H will be the required angle formed by A B and B C.

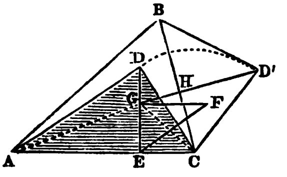


For A F is perpendicular to the plane C B C′; and B G F being a right angle in this plane, an oblique line drawn from A to G will be perpendicular to B G (7). The two right angled triangles B G H, B G A, having a common base B G and equal hypotenuses B H, B A, have therefore the angle G B H equal to G B A.

Remark. The angle contained by A G, F G will determine the dihedral angle formed by the plane A C and the projecting plane B C C′ (15); and that angle may be found, by constructing on the base A F a right angled triangle, of which the perpendicular is equal to F G.

PROBLEM XI.

118. *In a trihedral angle of which two faces and their included dihedral angle are given, to determine the remaining face.*



Place the given faces A B C, A D C in the same plane, and draw D E perpendicular to A C. At E make the angle D E F equal to the given dihedral angle; take E F equal to D E, and draw F G parallel to A C. From G draw G H D′ perpendicular to B C, and from C with the radius C D describe an arc intersecting G H D′ in the point D′; join C D′ and B C D′ will be the required face.

For the point G is the projection of the vertex D of the face A D C, elevated above the plane A B C; and G H being also the projection of H D′, the equal edges C D, C D′ will coincide; and B C D is therefore a face of the trihedral angle A B C D.

Remark. The dihedral angle formed by A B C, B C D′ will be determined by the angle at the base G H of a right angled triangle, of which G F is the perpendicular and H D′ the hypotenuse; and the inclinations of the edges A D, C D to the face A B C, will be similar angles corresponding to the projections A G, C G, and the common perpendicular F G.

PROBLEM XII.

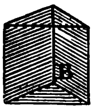
119. *To determine any of the dihedral angles in a trihedral angle, of which the faces are given.* (See fig. prob. XI.)

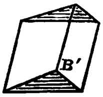
Place the given faces A B C, A D C, B C D′ in the same plane, and from the extremities D, D′ of the equal edges C D, C D′ draw D E, D′H G perpendicular to A C, B C. Through the point of intersection G, draw G F parallel to A C, and from E with the radius E D determine the point F. The angle F E G evidently corresponds to that required (118).

Remark. The dihedral angle formed by the faces A B C, B D′C has been already noticed; and a perpendicular drawn from G to A B together with F G, will make known the dihedral angle formed by the face A B C, and a plane passing through the points A, B, D.

PROBLEM XIII.

120. *To construct a triangular prism of which the base and two of the bounding sides are given.*



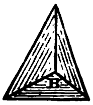


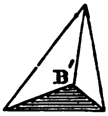
If the sides of the prism be rectangles, they will evidently be perpendicular to the base, and the construction is obvious (109). But if the sides be parallelograms, let the projection of their common edge be found by the construction of the preceding problem, and also the corresponding perpendicular to the base. The position of this edge may then be determined by prob. II., and the prism may be constructed by prob. I.

Remark. A similar construction is applicable to a quadrilateral and a polygonal prism, of which the base and any two adjacent parallelograms are given.

PROBLEM XIV.

121. *To construct a triangular pyramid of which the base and two of the bounding triangles are given.*



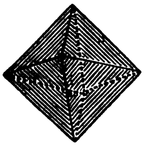


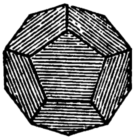
Find the projection of the edge common to the given sides of the pyramid, and also the corresponding perpendicular, by the construction of problem XII. The vertex of the pyramid being then determined by prob. II., the principal edges may be drawn, and the solid completed.

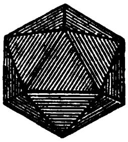
Remark. A similar construction is applicable to a quadrilateral or polygonal pyramid, of which the base and any two adjacent triangles are given.

PROBLEM XV.

122. *To construct a regular polyhedron of which one of the equal edges is given.*







First, let the regular solid be an octahedron (rem. 107). Describe a square on the given edge, and through its centre draw opposite perpendiculars, each equal to half its diagonal (109). Join the angular points of the square and the extremities of these perpendiculars, and the solid thus constructed will have its *eight* faces equilateral triangles, and its *six* tetrahedral angles respectively equal, and will therefore be a regular octahedron.

Secondly, let the solid be a dodecahedron. Construct a regular pentagon on the given edge (rem. 64, II), and let five equal pentagons meeting it and each other successively, form equal trihedral angles (119). Connect these with five equal contiguous pentagons, forming a sixth pentagon parallel to that constructed on the given edge. The solid thus enclosed evidently has *twelve* equal pentagonal faces, and *twenty* equal trihedral angles, and is therefore a regular dodecahedron.

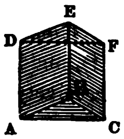
Thirdly, let the regular solid be an icosahedron. Construct a pyramid of which the base is a regular pentagon on the given edge, and its faces equilateral triangles. Find the dihedral angle formed by the faces of any of the equal trihedral angles of the pyramid (119), and at the vertices of the pentagon construct, by virtue of this dihedral angle, five equal pentahedral angles, of which the faces are equilateral triangles. The ten triangles thus joined to the faces of the pyramid will terminate in a regular pentagon equal to its base, and on which an equal pyramid being constructed, there will be formed a solid having *twenty* equal faces and *twelve* equal pentahedral angles, and which is a regular icosahedron.

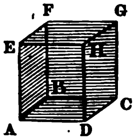
Remark. The number of tetrahedral angles of an octahedron is equal to the number of faces of a cube, and the number of trihedral angles of this solid is equal to the number of faces of the octahedron. It follows therefore that either of these regular solids may be inscribed in the other, so that its several vertices may be at the centres of the faces of the solid in which it is inscribed; and the same may be said of the dodecahedron and icosahedron, of which the number of faces and that of their trihedral or pentahedral angles are reciprocally related.

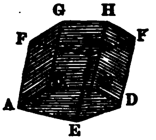
§ 2. OF MENSURATION RELATIVE TO PLANE SOLIDS.

PROBLEM I.

123. *To find the volume, or solid content, of a prism, of which the altitude and the area of the base are given.*







The volume$=ab$, ... [1],

in which a denotes the altitude and b the area of the base.

Examples.

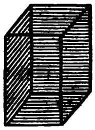
First, let $a=20$ feet, and let the base be an equilateral triangle, of which the side s is 10 feet; then $b=\frac{1}{4}s^2\sqrt{3}=$ 43.301 square feet (120, I); and the volume of the corresponding triangular prism is 866.02 cubic feet.

Secondly, let $a=12$ feet, and let the base be a parallelogram, of which the corresponding altitude and base are $a'=2$, $b'=4$ feet; the area $b=a'b'=8$ square feet (123, I), and the volume of the corresponding quadrilateral prism or parallelopiped is $12.8=96$ cubic feet.

Thirdly, let $a=6$ yards, and let the base be a regular pentagon, of which the side $s=10$ yards Then $b=$ $1.720477\,s^2=172.0477$ square yards (rem. 79, II); and the volume of the corresponding pentagonal prism is 1032.2862 cubic yards.

PROBLEM II.

124. *To find the volume of a rectangular parallelopiped of which the altitude and sides of the base are given.*

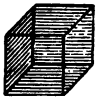


The volume$=abc$, ... [2], in which a is the altitude, and b, c are the adjacent sides of the rectangular base.

Example. Let $a=12$, $b=4$, $c=2$ yards; then the volume is 96 cubic yards.

PROBLEM III.

125. *To find the volume of a cube of which the side of the base is given.*



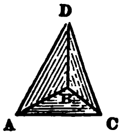
The volume$=s^3$, ... [3],

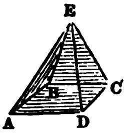
in which s is the given side.

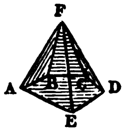
Example. Let $s=12$ feet. The volume is $12^3=1728$ cubic feet.

PROBLEM IV.

126. *To find the volume of a pyramid of which the altitude and the area of the base are given.*







The volume$=\frac{1}{3}ab$, ... [4],

in which a is the altitude and b the area of the base.

Examples.

First, let $a=6$ yards, and let the base be a triangle of which the sides are $a'=6$, $b'=12$, $c'=8$ feet; then $b=\sqrt{(13.7.1.5)}=21.31$ square feet (122, I), and the volume of the corresponding triangular pyramid is 127.86 cubic feet.

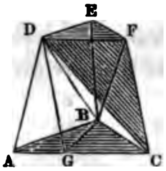
Secondly, let $a=8$ feet, and let the base be a quadrilateral capable of being inscribed in a circle, and of which the sides are $a'=3$, $b'=4$, $c'=5$, $d'=6$ feet; then $b=18.97$ square feet (77, II), and the volume of the corresponding quadrilateral pyramid is 50.586 cubic feet.

Thirdly, let $a=5$ feet, and let the base be a regular hexagon of which the side $s=4$ feet; then $b=\frac{3}{2}r^2\surd 3=24\surd 3$ square feet (73, 81, II); and the volume of the corresponding hexagonal pyramid is $40\surd 3=69.282$ cubic feet.

Fourthly, let the pyramid be regular, and let the side of its equilateral base or side be $s=6$ yards. In this case, $a=\surd(s^2-r^2)$, in which r is the radius of a circle circumscribing the base, $b=\frac{3}{4}r^2\surd 3$, and $s=r\surd 3$ (74, 73, II); wherefore the volume of the regular pyramid or tetrahedron is $\frac{1}{4}r^3\surd 6=\frac{1}{12}s^3\surd 2=25.45578$, according to the preceding volume of s in terms of the radius r.

PROBLEM V.

127. *To find the volume of the frustum of a pyramid, of which the altitude and the areas of the two bases are given.*



$$\text{The volume}=\tfrac{1}{3}a\,\{b+b'+\surd(b\,b')\},\ \ldots\ [5],$$

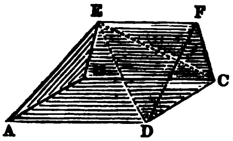
in which a denotes the altitude of the frustum, and b, b' are the areas of its similar triangular, quadrilateral, or polygonal bases.

Example. Let $a=6$ feet and $b=9$, $b'=4$ square feet; then $\surd(b\,b')=6$, and the volume of the corresponding frustum is 38 cubic feet.

Remark. Should a and b be given, and also the altitude a' of the entire pyramid of which the frustum constitutes a part, the upper base b' may be found by theorem VIII., and thence the volume of the frustum by the preceding formula. The frustum may also be considered as the difference of two similar pyramids, and its volume will become known by means of problem IV. and theorem XIV.

PROBLEM VI.

128. *To find the volume of a rectangular pyramoid or wedge, of which the altitude, base, and parallel edge are given.*



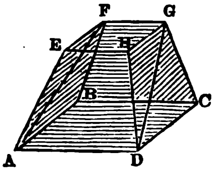
The volume$=\frac{1}{6}ab(2c+c')$, ... [6],
in which a is the altitude; b, c are the adjacent sides of the base, and c' is the edge parallel to the side c.

For a diagonal plane E C D divides the pyramoid A C E F into the quadrilateral pyramid A B C D E and the triangular pyramid C F D E, the volumes of which being $\frac{1}{3}abc$ and $\frac{1}{6}abc'$, there results formula [6].

Example. Let $a=10$ feet, $b=20$, $c=18$, and $c'=12$; the volume is 1600 cubic feet.

PROBLEM VII.

129. *To find the volume of the frustum of a rectangular pyramoid cut by a plane parallel to the base, having given the altitude and sides of the bases.*



The volume$=\frac{1}{6}a\{b(2c+c')+b'(2c'+c)\}$, . . [7],
in which a is the altitude and b, c; b', c' are the corresponding sides of the parallel bases.

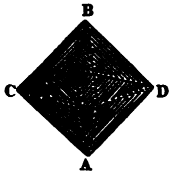
For a diagonal plane A F G D evidently divides the frustum A C E G into two rectangular pyramoids A C F G and E G A D, the volumes of which constitute formula [7].

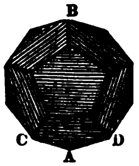
Example. Let $a=12$ feet; $b=20$, $c=10$; $b'=16$, $c'=5$; the volume is $2(20.25+16.20)=1640$ cubic feet.

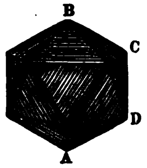
Remark. The frustum here considered is frequently called a *rectangular prismoid.*

PROBLEM VIII.

180. *To find the volume of a regular polyhedron of which one of the equal edges is given.*







The volume $=\frac{1}{3}nab$, ... [8],

in which n is the number of equal triangular pyramids into which the polyhedron may be divided; a the altitude of these pyramids, and b the area of any of their equal bases.

Examples.

First, let the solid be a regular octahedron of which the given edge is c. This solid may be divided into two quadrilateral pyramids, of which the base is a square formed by any four edges in the same plane, and the altitude is evidently half the diagonal of an equal square; wherefore $b=c^2$, $a=\frac{1}{2}c\sqrt{2}$, $n=2$, and the volume of the octahedron is $\frac{1}{3}c^3\sqrt{2}=.4714\,c^3$.

Secondly, let the regular solid be a dodecahedron, which may be divided into twelve equal pentagonal pyramids, having a common vertex at a central point O within the solid. Through this point and two diametrically opposite vertices A, B conceive a right line A B to be drawn, and let similar right lines join each of these points, and two vertices C, D adjacent to A in the lowest pentagonal face of the dodecahedron. Put for the moment $AB=2r$, $BC=c'$, $CD=c''$, and let r' be the radius of a circle described about the face corresponding to the points A, C, D. Then A C B being the angle in a semicircle of which O is the centre, $2r=\sqrt{(c^2+c'^2)}$; and c, c'' being the sides of regular pentagons of which c'', c' are evidently diagonals, $c=\frac{1}{2}c''(\sqrt{5}-1)$, $c''=\frac{1}{2}c'\{\sqrt{5}-1\}$ (cor. 32, 78, II); whence $c'=\frac{1}{2}c(3+\sqrt{5})$ and $r=c\sqrt{(\frac{9+3\sqrt{5}}{8})}$. But the radius $r'=c\sqrt{\{\frac{5+\sqrt{5}}{10}\}}$ (prob. VI. VII); the altitude from O to the corresponding pentagon is $a=\sqrt{(r^2-r'^2)}=\frac{1}{2}c\sqrt{(\frac{25+11\sqrt{5}}{10})}$; the area of this pentagon is $b=\frac{1}{4}c^2\sqrt{\{25+10\sqrt{5}\}}$ (rem. 79, II), and $n=12$; wherefore the volume of the regular dodecahedron is $\frac{5}{2}c^3\sqrt{(\frac{47+21\sqrt{5}}{10})}=7.66312\,c^3$.

Lastly, let the solid be a regular icosahedron, considered as formed of twenty equal triangular pyramids, having their vertex at a central point O. Conceive a right line A B to be drawn as before; and draw the diagonal A C of the regular pentagon of which three of the five adjacent vertices are A, C, D. Put $AB=2r$, $AC=c'$, and let r' be the radius of any of the equilateral faces of the icosahedron. Then A C B being a right angle, $2r=\sqrt{(c^2+c'^2)}$, and c' being the diagonal of a regular pentagon of which c is the side, $c=\frac{1}{2}c'(\sqrt{5}-1)$, or $c'=\frac{1}{2}c(1+\sqrt{5})$; wherefore $r=c\sqrt{(\frac{5+\sqrt{5}}{8})}$. But $r'=\frac{c}{\sqrt{3}}$ (prob. V. II), $a=\sqrt{(r^2-r'^2)}=\frac{c}{2}\sqrt{(\frac{7+3\sqrt{5}}{6})}$, $b=\frac{1}{4}c^2\sqrt{3}$ (prob. IV. I), and $n=20$; wherefore the volume of the icosahedron is $\frac{5}{6}c^3\sqrt{(\frac{7+3\sqrt{5}}{2})}=2.1817\,c^3$.

Remark. The value of O A drawn from the central point O to any of the angles of the dodecahedron is $\frac{1}{2}c'\sqrt{3}$, which expresses half the diagonal of an inscribed cube having its side c' equal to the diagonal of one of the pentagonal faces of the dodecahedron.

BOOK IV.

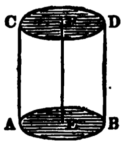
OF CURVE OR ROUND SOLIDS.

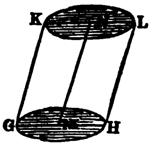
CHAPTER I.

OF THE CYLINDER.

1. When a right line in contact with the circumferences of two equal and parallel circles, moves round them so as to be parallel to a right line joining their centres, it describes a curve surface enclosing in connexion with these circles a portion of space, and constituting with them a curve surface, which is characterized by the regularity and positions of its boundaries.

2. A *cylinder* is a curve or round solid, bounded by two equal and parallel circles and by a curve surface, such that a right line may be drawn in it from the circumference of either circle, parallel to a right line joining the centres of the circles. The equal and parallel circles are the *bases* of the cylinder; the right line which joins the centres of these bases is termed its *axis*.





The solid A B C D represents a cylinder, of which A B, C D are the bases, and E F is its axis. The solid G H K L is also a cylinder; G H, K L are the bases, and M N the axis.

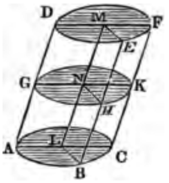
3. A cylinder takes denominations relative to the position of its axis, which may be perpendicular or oblique to its bases. A *right* cylinder has its axis perpendicular to its bases, as A B C D; an *oblique* cylinder has its axis oblique to its base, as the solid G H K L.

Remark. A right cylinder may evidently be generated by the revolution of a rectangle about one of its sides considered as fixed; and may therefore be denominated a solid of *revolution.*

OF THE SECTIONS OF A CYLINDER BY A PLANE.

THEOREM I.

4. *If a cylinder be cut by a plane parallel to the base, the section is a circle equal to the base.*



Let A C D F be a cylinder, in which G H K is a section parallel to the base A B C; this section is a circle equal to A B C.

Draw the axis L M, meeting the plane of G H K in N; join G N, H N, and let planes passing through these lines and L N, intersect the curve surface and the base of the cylinder in the right lines A G, A L and B H, B L. Because the plane A N cuts the parallel planes A C, G K, the sections A L, G N are parallel (21); and because A G, L N are also parallel (2), the figure A L G N is a parallelogram, and therefore G N is equal to A L. In a similar manner, it may be proved that H N is equal to B L or A L, and the same may be said of any other right line N K drawn from N to the section G H K; wherefore this section is a circle equal to the base A B C.

Remark. The cylinder A C D F being considered as oblique, another section equal but inclined to G H K, may be formed by a plane drawn perpendicular to the plane which projects the axis L M on the base A B C (115, III). This section makes with one of the parallel intersections of the curve surface and the projecting plane, an angle equal to that formed by the base and the other intersection; it is called a *subcontrary* section of the cylinder.

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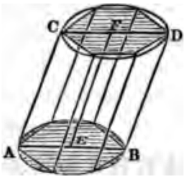
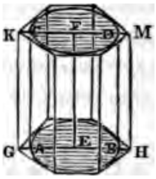
§ 2. OF PRISMS DESCRIBED IN AND ABOUT CYLINDERS.

5. When the bases of a prism are inscribed in the bases of a cylinder, the *prism* is said to be *inscribed in the cylinder;* and when the bases of a prism are described about the bases of a cylinder, the *prism* is said to be *described about the cylinder.*





Of the preceding solids, A B C D is a prism inscribed in the cylinder A D; and G H K M is a prism described about, or circumscribing, the cylinder A B C D.

6. Two prisms which have similar bases, and which are described in or about *similar* cylinders, having their axes equally inclined to their bases, and proportional to the radii of these bases, are evidently equiangular and similar. The comparison of these prisms as respects their areas and their volumes may be instituted by means of the two following theorems:

THEOREM II.

7. *The convex surfaces of similar prisms described in or about cylinders, are proportional to the squares of the radii of the bases of those cylinders.* (See fig. the. IV.)

For the several corresponding faces which constitute the convex surfaces of the prisms, being similar parallelograms, are proportional to the squares of their edges (88, I); and these edges being respectively proportional to the radii of the bases of the cylinders (35, II), the convex surfaces of the inscribed or circumscribed prisms are proportional to the squares of the radii of the bases of the cylinders.

Corollary. The convex surfaces of the inscribed or circumscribed prisms are also proportional to the bases of the cylinders.

THEOREM III.

8. *The volumes of similar prisms described in or about cylinders, are proportional to the cubes of the radii of the bases of those cylinders.*

For the volumes of similar prisms are proportional to the cubes of their edges (73, III); and these edges being respectively proportional to the radii of the bases of the cylinders (35, II), the volumes of the prisms are proportional to the cubes of the radii of the bases of the cylinders.

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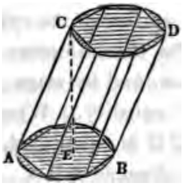
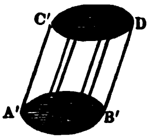
§ 3. OF THE AREAS OF THE CURVE SURFACES OF CYLINDERS.

THEOREM IV.

9. *The curve surfaces of similar cylinders are proportional to the squares of the radii of their bases.*





For if regular and similar polygonal prisms be inscribed in the cylinders A B C D, A'B'C'D', their convex surfaces will be proportional to the squares of the radii A E, A'E' (7); and this proportionality will subsist whatever be the number of faces bounding the prisms. But if this number be continually increased, the corresponding surfaces will also increase, and will continually approach the curve surfaces of A B C D, A'B'C'D', which form their respective limits of the cylinder A B C D is equal to the product of the altitude C E and the area of the base A B A.

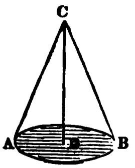
Corollary. The volume of a cylinder is equal to that of a prism having an equal base and the same altitude.

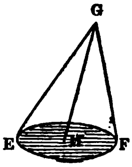
CHAPTER II.

OF THE CONE.

13. When a right line passing through a point given without the plane of a circle, moves round the corresponding circumference, it describes a curve surface enclosing, in connexion with this circle, a portion of space, and constituting with it a curve solid which is characterized by the regularity and positions of its boundaries.

14. A *cone* is a curve or round solid bounded by a circle, and by a curve surface which terminates in a point, and which is such that a right line may be drawn in it from this point to the circumference of the circle. The bounding circle is the *base* of the cone; the point in which the curved surface terminates is the *vertex;* any right line drawn in the surface from the vertex is called the *slant side* of the cone, and the right line joining the vertex and the centre of the base is termed its *axis*.





The solid A B C represents a cone, of which A B is the base; C is the vertex; A C or B C a slant side; and C D the axis. The solid E F G is also a cone; E F is its base; G the vertex; E G or F G a slant side; G H the axis.

15. A cone takes denominations relative to the position of its axis, which may be perpendicular or oblique to its

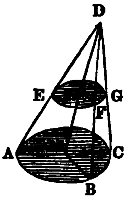
base. A *right* cone has its axis perpendicular to the base, as A B C; an *oblique* cone has its axis oblique to its base, as the solid E F G.

Remark. A right cone may evidently be generated by the revolution of a right angled triangle about either of its perpendicular sides considered as fixed; and may therefore be denominated a solid of *revolution*.

§ 1. OF THE SECTIONS OF A CONE BY A PLANE.

THEOREM I.

16. *If a cone be cut by a plane parallel to the base, the section is a circle.*



Let A B C D be a cone, and E F G a section parallel to the base A B C; this section is a circle.

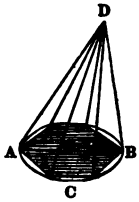
Draw the axis D H, meeting E F G in K; join E K, F K, and let planes passing through these lines and D H intersect the curved surface and base of the cone in the right lines A D, A H and B D, B H. Because the plane A K cuts the parallel planes A C, E G, the sections A H, E K are parallel (21); and the triangles A D H, E D K being therefore equiangular, A H : E K :: D H : D K (49, I). For a like reason B H : F K :: D H : D K, which compared to the preceding analogy gives A H : B H :: E K : F K. But the base A B C being a circle, A H is equal to B H; wherefore E K is equal to F K; and the same may be said of any other right line K G drawn from K to the section E F G, which is therefore a circle.

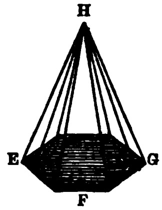
Remark. The cone being considered as oblique, another section similar but inclined to E F G may be formed by a plane drawn perpendicular to the plane which projects the axis DH on the base ABC (115). This section makes with one of the slant sides in the projecting plane, an angle equal to that formed by the base and the other slant side; it is called a *subcontrary* section of the cone.

---

§ 2. OF PYRAMIDS DESCRIBED IN AND ABOUT CONES.

17. When the base of a pyramid is inscribed in the base of a cone, and both these solids have the same vertex, the *pyramid* is said to be *inscribed in the cone;* and when the base of a pyramid is described about the base of a cone, and these solids have the same vertex, the *pyramid* is said to be *described about the cone.*





Of the preceding solids, A B C D is a pyramid inscribed in the cone A C D, and E F G H is a pyramid described about the cone A B H.

18. Two pyramids which have similar bases, and which are described in or about *similar* cones, having their axes equally inclined to their bases, and proportional to the radii of these bases, are evidently equiangular and similar. The comparison of these pyramids as respects their areas and their volumes may be instituted by means of the two following theorems:

THEOREM II.

19. *The convex surfaces of similar pyramids described in or about cones are proportional to the squares of the radii of the bases of those cones.* (See fig. the. IV.)

For the several corresponding faces which constitute the convex surfaces of the pyramids being similar triangles, are proportional to the squares of their edges (70, I); and these edges are proportional to the radii of the bases of the cones (35, II); wherefore the convex surfaces of the inscribed or circumscribed pyramids are proportional to the squares of the radii of the bases of the cones.

Corollary. The convex surfaces of the inscribed or circumscribed pyramids are also proportional to the bases of the cones.

THEOREM III.

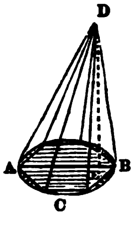
20. *The volumes of similar pyramids described in or about cones, are proportional to the cubes of the radii of the bases of those cones.*

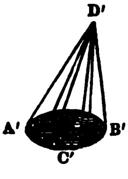
For the volumes of similar pyramids are proportional to the cubes of their edges (98, III), and these edges are respectively proportional to the radii of the bases of the cones (35, II); wherefore the volumes of the inscribed or circumscribed pyramids are proportional to the cubes of the radii of the bases of the cones.

§ 3. OF THE AREAS OF THE CURVED SURFACES OF CONES.

THEOREM IV.

21. *The curve surfaces of similar cones are proportional to the squares of the radii of their bases.*



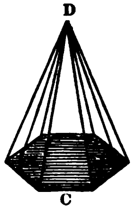


For if regular and similar polygonal pyramids be inscribed in the cones A B C D, A'B'C'D', their convex surfaces will be proportional to the squares of the radii A E, A'E'; and this proportionality will subsist whatever be the number of faces bounding the pyramids. But if this number be continually increased, the corresponding surfaces will also increase, and will continually approach the curve surfaces A B C D, A′B′C′D′, which form their respective limits, and which must have their invariable ratio $AE^2 : A'E'^2$. Wherefore the area of A B C D : the area of A′B′C′D′ :: $AE^2 : A'E'^2$.

Corollary. The curve surfaces of similar cones are proportional to the squares of their axes, or their altitudes; and when these solids are right cones, their curve surfaces are also proportional to the squares of their slant sides.

THEOREM V.

22. *The curve surface of a right cone is equal to the product of its slant side and the semicircumference of its base.*



Let A B C D be a right cone of which the slant side is A D; the area of the curve surface of A B C D is equal to the product of A D, and the semicircumference A B C.

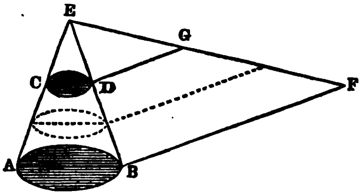
For if a regular polygonal pyramid be described about the cone A B C D, the area of its convex surface, formed of equal triangles, will evidently be equal to the product of the slant side A D and the semiperimeter of its polygonal base; and this area and corresponding product, by increasing the number of sides of the pyramid, will continually approach the area of the curve surface of the cone, and the product of its slant side, and the semicircumference of its base, which form their respective limits, and between which there must subsist a corresponding equality. Wherefore the curve surface of A B C D is equal to the product of the slant side A D and the semicircumference A B C.

Corollary. The area of the curve surface of a right cone is equal to the area of a right angled triangle, of which the perpendicular sides are respectively equal to the slant side of the cone and the circumference of the base.

Remark. The curve surface of a cone may be unfolded on a plane; and when this solid is a right cone, the corresponding *development* of its curved surface is evidently a circular sector, of which the radius and bounding arc correspond to the slant side and the circumference of the base of the cone.

THEOREM VI.

23. *The curve surface of the frustum of a right cone is equal to the product of its slant side, and half the sum of the circumferences of its bases.*



Let A B C D be the frustum of a right cone, of which the slant side is B D; the curve surface of A B C D is equal to the product of B D, and half the circumferences A B A, C D C.

Produce A C, B D until they meet in E, the vertex of the entire cone A B E, of which the cone C D E is a part and also the frustum A B C D. From the points B, D draw the right lines B F, D G perpendicular to B E; let B F be equal to the circumference A B A, and draw the right line E F, meeting D G in the point G. Because the triangles A E B, C E D are similar, and also the triangles B E F, D E G, the sides B E, D E are proportional to A B, C D and also to B F, D G (50, I); wherefore A B : C D :: B F : D G. But the radii A B, C D are proportional to the circumferences A B A, C D C (44, II), and B F is equal to the circumference A B A; wherefore D G is equal to the circumference C D C. Now the curve surface of the entire cone A B E is equal to the area of the right angled triangle E B F, and the curve surface of the part or cone C D E is equal to the area of the triangle E D G (cor. 22); wherefore the remaining curve surface of the frustum A B C D is equal to the area of the trapezoid B D G F, which is equal to the product of B D and half the sum of B F, D G, or of the circumferences A B A, C D C (124, I).

Corollary. The curve surface of the frustum of a right cone is equal to the product of its slant side, and the circumference of a section made by a plane, drawn through the middle of this side, parallel to the bases of the frustum.

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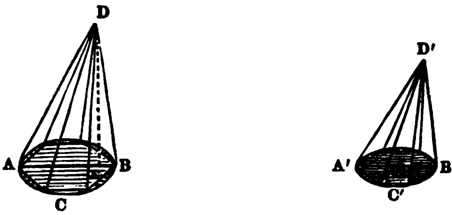
§ 4. OF THE VOLUMES OF CONES.

THEOREM VII.

24. *The volumes of similar cones are proportional to the cubes of the radii of their bases.*



For if regular and similar polygonal pyramids be described in or about the cones A B C D, A′B′C′D′, their volumes will be proportional to the cubes of the radii A E, A′E′, whatever be the number of their faces and however near they may approach the cones A B C D, A′B′C′D′ which form their respective limits, and which must have their invariable ratio $AE^3 : A'E'^3$. Wherefore the volume of A B C D : the volume of A′B′C′D′ :: $AE^3 : A'E'^3$.

Corollary. The volumes of similar cones are proportional to the cubes of their axes or their altitudes; and when these solids are right cones, their volumes are also proportional to the cubes of their slant sides.

THEOREM VIII.

25. *The volume of a cone is equal to a third part of the product of its altitude and the area of its base.* (See fig. the. VII.)

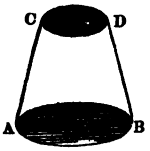
Let A B C D be a cone of which D E is the altitude; the volume of A B C D is equal to a third part of the product of D E and the area of the base A B C.

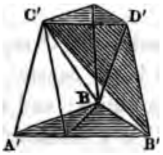
For if a regular polygonal pyramid be inscribed in the cone A B C D, its volume will be equal to a third part of the product of the altitude D E and the area of its polygonal base; and this volume and corresponding product, by increasing the number of faces of the pyramid, will continually approach the volume of the cone, and a third part of the product of its altitude and the area of its base, which form their respective limits, and between which there must subsist a corresponding equality. Wherefore the volume of the cone A B C D is equal to a third part of the product of the altitude D E and the area of the base A B C.

Corollary. The volume of a cone is equal to the volume of a pyramid, having an equal base and altitude, and is also equal to a third part of the volume of a cylinder, having the same base and altitude.

THEOREM IX.

26. *The volume of the frustum of a cone is equal to the volumes of three cones having the same altitude as the frustum, and their bases the upper and lower bases of this solid and a mean proportional between these bases.*





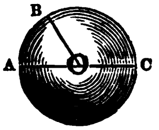
For the volume of the frustum of a cone A B C D is evidently equal to the volume of the frustum of a pyramid A′B′C′D′, having its altitude and bases respectively equal to those of the conic frustum (cor. 25); and the volumes of the three pyramids composing A′B′C′D′ being respectively equal to three cones having corresponding equal bases and altitudes (94), the volume of the conic frustum A B C D is equal to the volumes of those cones, of which the bases are AB, C D, and a mean proportional between them, and which have their common altitude the same as that of A B C D.

CHAPTER III.

OF THE SPHERE.

27. When a semicircle revolves about its diameter considered as fixed, the corresponding semicircumference describes a curve surface enclosing a portion of space and constituting a round solid, which is characterized by its perfect regularity, and the uniformity of its boundary.

28. A *sphere* is a curve or round solid bounded by a curve surface, such that all right lines drawn to it from an interior point, are respectively equal. The interior point is the *centre* of the sphere; any of the equal right lines drawn from the centre to the curve surface is a *radius;* and any right line passing through the centre and terminating in the surface is a *diameter.*



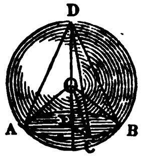
The solid A B C represents a sphere, of which the interior point O is the centre; any of the equal right lines O A, O B; O C is a radius; and the right line A C passing through the centre O is a diameter.

29. The position of a sphere is evidently determined by the position of its centre; and its magnitude depends on its radius, which may be equal to any given right line, or may extend to any required distance from the centre.

§ 1. OF THE SECTIONS OF A SPHERE BY A PLANE.

THEOREM I.

30. *If a sphere be cut by a plane, the section is a circle.*



Let A B C D be a sphere cut by a [illegible]; the section A B C is a circle.

The intersecting plane may pass through the centre O of the sphere, or this point may be situated without the plane. In the first case, the corresponding section is evidently a circle, having its radius O A the same as the radius of the sphere. In the second case, draw from the centre O to the plane of the section A B C the perpendicular O E, and join O, E, and any two points A, B in A B C. Because O E is perpendicular to the plane of A B C, the angles O E A, O E B are right angles (5, III); and because these angles have the side O E common and are subtended by the equal radii or hypotenuses O A, O B, the remaining sides E A, E B are respectively equal (40, I); and the same may be said of E A and any other right line E C drawn from E to the section A B C; wherefore the section A B C is a circle.

Corollary. The section or circle which corresponds to a plane drawn through O is greater than the section A B C; for O A is evidently greater than E A, the radius of this section.

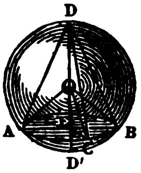
31. A *great circle* of the sphere is that formed by a plane, which passes through the centre of the sphere. This circle divides the sphere into parts which become identical when applied to each other, and which are therefore hemispheres, having equal surfaces for their curved boundaries. Two great circles of the sphere evidently pass through the same diameter, and therefore bisect each other. They enclose

equal and opposite portions of the surface, which are termed *spherical lunes.*

32. A *small circle* of the sphere is that formed by a plane which does not pass through the centre of the sphere. A small circle divides the sphere into unequal parts, each of which is a *spheric segment,* having this circle for its base. Two small circles at equal perpendicular distances from the centre of the sphere are evidently equal to each other: their circumferences enclose a portion of the spheric surface termed a *zone.*

THEOREM II.

33. *The circumference of any circle of the sphere is equally distant from either extremity of a diameter drawn perpendicular to that circle.*



Let A B C be the circumference of a circle of the sphere A B C D, in which D D′ is a diameter perpendicular to the plane of A B C; the circumference A B C is equally distant from D, the extremity of D D′

Take any two points A, C in the circumference A B C, of which the centre is E (cor. 30), and draw the chords A D, C D and the radii A E, C E. Because D E is perpendicular to the plane of A B C, the angles A E D, C E D are right angles (5, III); and because these angles have the side D E common and A E is equal to C E, the right angled triangles A D E, C D E are equal (38, I), and A D is therefore equal to C D. In like manner, it may be proved that A D is equal to any other right line B D drawn from the point D to the circumference A B C; wherefore this circumference is equally distant from the point D.

Corollary. The arcs of great circles between either extremity of a diameter of a sphere and the circumference of any circle to which this diameter is perpendicular, are equal to each other; for the chords which subtend those arcs are respectively equal.

Remark. The circumference A B C may be referred to the points D, D′ as its centres, and may be considered as determined, when either of these points, as D, and the arc A D of any great circle passing through it are given.

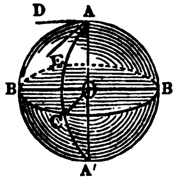
34. The *poles* of a circle of the sphere are two points in the surface from each of which the circumference of the circle is equally distant. The *axis* of a circle of the sphere is the diameter joining its poles.

Of the circle A B C, the poles are D, D′; and its axis is the diameter D D′, which joins these poles and which is perpendicular to A B C at its centre E.

Corollary. The corresponding poles of parallel circles are coincident; and the poles of a great circle are at quadrantal distances from its circumference. This great circle evidently bisects all great circles passing through its poles, and is perpendicular to each of them.

THEOREM III.

35. *The dihedral angle formed by the planes of two arcs of great circles of a sphere, is determined by the plane angle contained by tangents to those arcs at their point of meeting.*



Let A B, A C be arcs of great circles of the sphere A B A′B′, and let B A O C be the dihedral angle formed by their planes, and D A E the angle contained by their tangents A D, A E; the angle B A O C is determined by the angle D A E.

Because A D is a tangent to the arc A B, it is perpendicular to the radius A O (cor. 9, II); and for a like reason A E is also perpendicular to A O. But A O is the common section of the planes of the arcs A B, A C; wherefore the angle D A E, contained by the perpendiculars A D, A E to this common section, determines the dihedral angle B A O C (15, III). (See fig. the. IV.)

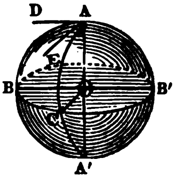
Remark. The angle D A E contained by the tangents A D, A E to the arcs A B, A C, is considered as equal to the angle formed by those arcs; for this angle B A C viewed independently of the lengths of the arcs A B, A C is the same as the angle D A E corresponding to their tangents.

36. A *spherical angle* is the opening formed by two arcs of great circles of a sphere. The point in which the arcs meet is the *vertex* of the spherical angle formed; the arcs are the *sides* of this angle.

The angle B A C is a spherical angle, of which A is the vertex, and A B, A C are its sides.

THEOREM IV.

37. *A spherical angle is measured by the arc of a great circle, which is intercepted by its sides, and of which the pole is the vertex of the angle.*



Let B A C be a spherical angle, and B C the arc of a great circle of which the pole is the vertex A; the angle B A C is measured by the arc B C.

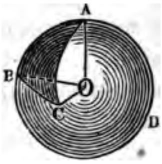
Draw the radii O B, O C, and the tangents D A, E A to the sides B A, C A of the spherical angle B A C. Because the point A is the pole of the arc B C, the radius A O is perpendicular to the plane O B C (34), and the angles A O B, A O C are right angles; wherefore the angle B O C determines the dihedral angle B A O C and is equal to the angle D A E contained by the tangents at A, and which corresponds to the spherical angle B A C. But the arc B C measures the angle B O C (cor. 43, II); wherefore this arc also measures the spherical angle B A C.

Corollary. A spherical angle is also measured by the arc of a great circle between the corresponding poles of its sides; for this arc and that just considered are equal parts of the same great circle, having its pole the vertex of the spherical angle.

Remark. Spherical angles are *right* or *oblique*, and their sides are *perpendicular* or *oblique* to each other, according as their corresponding plane and dihedral angles are right or oblique; or according as the arcs which measure them are equal to or different from quadrants; and *adjacent* and *vertical* spherical angles possess properties analogous to those considered in article 14, 15, 16, Book I.

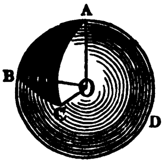
THEOREM V.

38. *The trihedral angle formed by the planes of three arcs of great circles of a sphere, is determined by those arcs and their corresponding spherical angles.*



Let A B, A C, B C be arcs of great circles of the sphere A B C D, and let A B C O be the trihedral angle formed by their planes; the angle A B C O is determined by the arcs A B, A C, B C and the spherical angles A, B, C.

For the plane angles A O B, A O C, B O C of the trihedral angle A B C O, are respectively measured by the subtended arcs A B, A C, B C having equal radii; and the dihedral angles contained by its faces are determined by the spherical angles at the extremities of the edges O A, O B, O C; wherefore the trihedral angle A B C O is determined by the arcs A B, A C, B C, and the corresponding spherical angles A, B, C.



Remark. The arcs A B, A C, B C enclose a portion of the surface of the sphere, which may be considered as the curve base of a spherical triangular pyramid, having its vertex the centre O of the sphere, and of which the faces are the circular sectors O A B, O A C, O B C.

§ 2. OF SPHERICAL TRIANGLES.

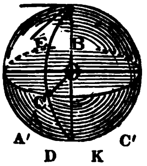
39. *A spherical triangle* is a part of the surface of a sphere contained by three arcs of great circles. The figure A B C represents a spherical triangle; the arcs A B, A C, B C are its *sides*, and the point B is its *vertex* opposite to the *base* A C.

40. A spherical triangle is *equilateral*, or *isosceles*, or *scalene*, according as all its sides are equal, or two of them are equal, or the three are unequal; and the angles opposite to those sides are equal or unequal, as in plane triangles considered in article 30, Book I.

THEOREM VI.

41. *The angles of any spherical triangle, having their vertices the poles of the sides of another spherical triangle, are measured by the supplements of these sides; and the angles of the latter triangle are measured by the supplements of the sides of the former triangle.*

Let A B C be a spherical triangle, of which the vertices A, B, C are the poles of B′C′, A′C′, A′B′, the sides of the spherical triangle A′B′C′; the angles A, B, C, are measured by the supplements of B′C′, A′C′, A′B′, and the angles A′, B′, C′, are measured by the supplements of the sides B C, A C, A B.



Produce the sides A B and A C both ways to meet the sides A′C′, B′C′ and A′B′, B′C′ in the points D, E and F, G. Because the arc B C passes through the poles of A′B′, A′C′, their intersection A′ is the pole of this arc (cor. 34), and for a like reason, B′, C′ are the poles of A C, A B; wherefore the arcs B′G, C′E are quadrants, and B′C′, E G are together equal to the semicircumference of a circle, and are therefore supplements of each other. But the arc E G is the measure of the spherical angle A (37); wherefore this angle is measured by the supplement of the side B′C′. In like manner, it may be proved that the angles B, C are measured by the supplements of the sides A′C′, A′B′ of the triangle A′B′C′.

Again, because the arcs A E, B D are quadrants, D E, A B are together equal to the semicircumference of a circle, and are supplementary arcs. But the spherical angle C′ is measured by the arc D E (37); wherefore this angle is measured by the supplement of the side A B; and in the same manner it may be proved that the angles B′, C′ are measured by the supplements of the sides A C, A B.

Remark. The spherical triangles A B C, A′B′ C′ having the angles of either measured by the supplements of the sides of the other, are called *supplemental* triangles; and these sides having their poles the vertices of the angles to which they are related as measures, the triangles A B C, A′B′C′ are also denominated *polar* triangles.

THEOREM VII.

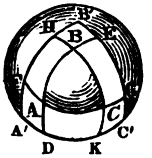
42. *Any two sides of a spherical triangle are greater than the third side; and the three sides are together less*

than four right angles, or the circumference of a great circle. (See fig. the. v.)

For the spherical triangle A B C being considered as the base of a spheric pyramid (rem. 38), of which the vertex is the centre O of the sphere A B C O, the three sides A B, B C, C A will measure the plane angles at this vertex, and must therefore have the properties of these angles in accordance with articles 26, 27, Book I.

THEOREM VIII.

43. *The three angles of every spherical triangle are together greater than two, and less than six, right angles.*



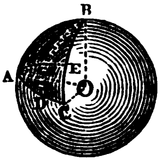
For the measures of the angles of the spherical triangle A B C being the supplements of the sides of its polar triangle A′B′C′, these measures and sides are together equal to six right angles. But the sides of the triangle A′B′C′ are together less than four right angles; wherefore the angles of A B C are together greater than two right angles. Again, each of the angles of the spherical triangle A B C is less than two right angles, since its corresponding dihedral angle is less than two right angles; wherefore the three angles of A B C are together less than six right angles.

Corollary. The angles of a spherical triangle may be *right*, or *obtuse*, or *acute* angles; and the exterior angle, formed by producing any of its sides, being the supplement of the interior adjacent angle, is less than both the interior opposite angles of the triangle.

Remark. The three preceding theorems involve the characteristic and fundamental properties of spherical triangles. The following comprise several properties common to these figures and plane triangles, between which there appears to subsist a considerable analogy, and which in reference to their angles are distinguished by the same denominations. See sections 1, 2, Chap. II. Book I.

THEOREM IX.

44. *If two sides of a triangle be equal, the opposite angles are equal, and conversely.*



For in the isosceles triangle A B C, of which the equal sides are A B, B C, let the arc B D of a great circle bisect the angle B, and let the two component triangles formed be the bases of spheric pyramids, having their common vertex O the centre of the sphere A B C O. These pyramids have evidently two plane angles, and the included dihedral angle of the one respectively equal to two plane angles and the included dihedral angle of the other, and have therefore the remaining plane and dihedral angles respectively equal (29, III). But these dihedral angles are the spherical angles of the triangle A B C, and which are therefore respectively equal. The converse may be demonstrated in a similar manner by means of a supplemental triangle A′B′C′.

Corollary. The arc of a great circle, which bisects the vertical angle of an isosceles spherical triangle, bisects the base, and is perpendicular to it.

Remark. This theorem has already been considered as evident. It is here introduced in consequence of the corollary resulting from its demonstration.

THEOREM X.

45. *In every spherical triangle, the greater side is opposite to the greater angle; and conversely.* (See the preceding fig.)

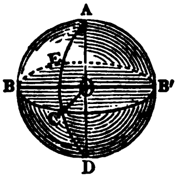
For let the greater angle A of the spherical triangle A B C be divided into two parts or angles B A E, C A E, by the arc A E of a great circle meeting B C in E; and let the angle

B A E be equal to the less angle B. The component triangle A B E being then isosceles, the sides A E, B E are equal (44), and therefore the side B C is equal to both the sides A E, C E of the other component triangle C A E. But these sides are together greater than the side A C (42); wherefore the side B C is greater than A C. The converse may be proved by indirect reasoning, as in (36, I), or by means of a supplemental triangle A′B′C′.

Remark. The preceding and its analogous theorem (36, I) have this peculiarity, that the part directly proved in one is indirectly proved in the other.

THEOREM XI.

46. *Any two sides of a spherical triangle are together equal to the semicircumference of a great circle, or greater, or less than it, according as the opposite angles are together equal to two right angles, or greater, or less, than two right angles.*



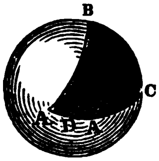
Produce the sides A B, A C of the spherical triangle A B C until they meet in D, making the arcs A B D, A C D semicircumferences of great circles, and forming the spherical angle B D C equal to B A C. The adjacent angles A B C, C B D being then together equal to two right angles, and the angles B D C, B A C equal to each other, the angle B D C will be equal to C B D, when the angles A B C, B A C are together equal to two right angles; and therefore in this case, the side B C is equal to C D (44). But A C, C D are together equal to the semicircumference A C D; wherefore the sides A C, B C are together equal to this semicircumference. In the second or third case of the theorem, the angle B D C will be greater or less than C B D; the side B C greater or less than C D (45); and the two sides A C, B C greater or less than the semicircumference A C D.

Remark. The converse may be proved in the same manner, or considered as a corollary by virtue of a supplemental triangle A′B′C′.

47. When two sides of a spherical triangle are both greater or both less than quadrants, they are said to be of the *same affection;* and when one of them is greater and the other less than a quadrant, they are said to be of *different affection.* In like manner, two angles are said to be of the same affection, when they are both greater or both less than right angles; and to be of different affection, when one of them is greater and the other less than a right angle.

THEOREM XII.

48. *The perpendicular sides of a right angled triangle are of the same affection with their opposite angles; and the hypotenuse is greater or less than a quadrant according as those sides are of different or the same affection.*



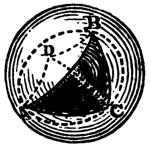
Let A B C be a right angled triangle of which C is the right angle; and in the side C A produced if necessary, take C D equal to a quadrant, and join B D by an arc of a great circle. Because the side or arc B C is perpendicular to the quadrant C D, the point D is its pole, and the arc B D is a quadrant perpendicular to B C (cor. 34); wherefore C B D is a right angle, and according as the angle A B C is greater or less than it, the opposite side A C is evidently greater or less than the quadrant C D; and conversely.

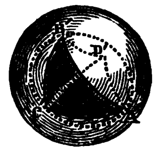
As to the second part of the theorem, the angles B D C, B A C will be greater or less than right angles, when the opposite side B C is greater or less than a quadrant; and the side A B will be greater or less than the quadrant B D, according as the angle B D A is greater or less than B A D (45), and the sides A B, B C are of different or of the same affection.

49. Two spherical triangles and their corresponding supplemental triangles being regarded as the curve bases of spheric pyramids, the equality of their sides and angles may be immediately established by articles 28, 29, Book III, and will be sufficiently evident without further notice. As to the surfaces of these figures, the three following theorems respect their equality and their most remarkable relations.

THEOREM XIII.

50. *The surfaces of two spherical triangles which have the three sides of the one respectively equal to the three sides of the other, are respectively equal.*





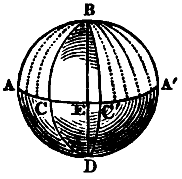
Let A B C, A′B′C′ be spherical triangles of which the sides A B, B C, A C are respectively equal to A′B′, B′C′, A′C′; the surface of A B C is equal to the surface of A′B′C′.

Through the vertices A, B, C and A′, B′, C′ describe small circles A B C, A′B′C′, of which the poles are D, D′, and join these poles and vertices by arcs of great circles A D, B D, C D and A′D′, B′D′, C′D′. Because the sides of the spherical triangles A B C, A′B′C′ are arcs of equal great circles, the chords corresponding to the equal sides are also equal; and therefore the circles A B C, A′B′C′, which circumscribe the plane triangles formed by these chords, are equal; and because the circumferences of equal circles are equally distant from their poles, the several arcs A D, B D, &c. are equal, and therefore the isosceles spherical triangles A D B, B D C, A D C will evidently coincide with and be respectively equal to the corresponding triangles A′D′B′, B′D′C′, A′D′C′. But the spherical triangle A B C is equal to the three triangles A D B, B D C, A D C, or to two of them diminished by the third, according to the interior or exterior position of the pole D; and the like may be said of the spherical triangle A′B′C′, as respects the triangles A′D′B′, B′D′C′, A′D′C′ and the pole D′; wherefore the surfaces of A B C, A′B′C′ being alike constituted of corresponding equal triangles, are respectively equal.

Remark. Should the spherical triangles A B C, A′B′C′ have their corresponding sides similarly situated, the equality of their surfaces may be proved by the coincidence of spheric pyramids, in accordance with article 30, Book III. But when these sides are dissimilarly situated, such coincidence will not take place, and the equality of the *reversed spherical triangles* A B C, A′B′C′ cannot be established by superposition.

THEOREM XIV.

51. *The surfaces of isosceles spherical triangles which have their equal sides quadrants, are respectively proportional to their bases, or to their corresponding vertical angles.*



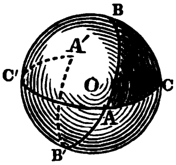
Let A B C, A′B C′ be two isosceles spherical triangles having their equal sides A B, B C and A′ B, B C′ quadrants; the surface of A B C : the surface of A′B C′ :: A C : A′ C′ :: the angle A B C : the angle A′B C′.

For if the bases A C, A′C′ contain the same arc or unit, three and five times respectively, so that A C : A′C′ :: 3 : 5; and if quadrants be drawn from the common vertex B to the several points of division in A C, A′ C′, the triangles A B C, A′ B C′ will be divided respectively into three and five equal parts, and therefore the surface of A B C : the surface of A′B C′ :: 3 : 5. But by hypothesis A C : A′ C′ :: 3 : 5, and the spherical angles A B C, A′ B C′ have the same ratio as the arcs A C, A′C′ (37); wherefore the surface of A B C : the surface of A′ B C′ :: A C : A′ C′ :: the angle A B C : the angle A′ B C′.

Corollary. The surfaces of the lunes B A D C B, B A′D C′B, formed by producing the sides B A, B C and B A′, B C′, are also proportional to the angles A B C, A′ B C′ at their vertices, since they are respectively twice the surfaces of the triangles A B C, A′ B C′; and if either of these lunes, as for instance B A D C B be compared to B A D E B taken equal to one fourth the surface of the sphere, and having the angle A B D a right angle, it will evidently follow that the surface of the lune B A D C B : one fourth the surface of the sphere :: the angle A B C : a right angle; in which proportion, the second and fourth terms may, when necessary, be changed into their multiples corresponding to the surface of the hemisphere or sphere, and to two or four right angles

THEOREM XV.

52. *The surface of any spherical triangle has to the surface of the sphere, the same ratio that the excess of the three angles of the triangle above two right angles, has to eight right angles.*



Let A B C be a spherical triangle, and let E denote the excess of the sum of its angles A, B, C above two right angles; the surface of A B C : the surface of the sphere :: the excess E : eight right angles.

Produce the side B C so as to form the great circle B C B′C′, bounding the hemisphere A B C O, on which the triangle A B C is situated; and let the other sides B A, C A produced, meet this circle in the points B′, C′, and also meet each other in the point A′, beneath the circle B C B′ C′ and in the surface of the hemisphere A′B′C′O. Because the arcs B A B′A′, C A C′A′ meet the circle B C B′ C′ in B′, C′, the arcs B C B′, C B′C′ are semicircumferences, and taking from each the common arc B′C, there remains the arc B C equal to B′ C′. In like manner it follows that the arcs A B, A C are respectively equal to A′B′, A′C′; and the spherical triangle A B C is therefore equal to the reversed spherical triangle A′B′C′ (50). Now the lune A B′A′ C′A being composed of the two spherical triangles A′B′C′, A B′ C′ on the hemispheres A′ B′ C′ O and A B C O, is therefore equal to the spherical triangles A B C, A B′C′ on the hemisphere A B C O; the lune B A B′C B, is equal to the triangles A B C, A B′ C, and the lune C A C′B C is also equal to the triangles A B C, A B C′ on the same hemisphere; wherefore the proportions which correspond to these lunes and the spherical angles A, B, C at their vertices are by cor. 51, the three following:

First, the lune A B′A′C′A, or the surfaces of A B C, A B′C′: the surface of A B C O :: the angle A : two right angles;

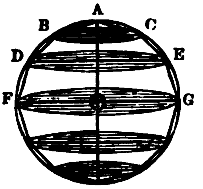
Secondly, the lune B A B′C B, or the sur. of A B C, A B′C : the surface of A B C O :: the angle B : two right angles;

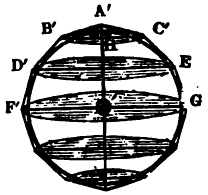
Thirdly, the lune C A C′B C, or the sur. of A B C, A B C′ : the surface of A B C O :: the angle C : two right angles; from which there results by composition and division of ratios, according to algebraical theorems 6 and 8,

the sur. of 3 A B C, A B′ C′, A B′ C, A B C′—sur. of A B C O : the sur. of A B C O :: the angles A, B, C —two right angles : two right angles. But the four spherical triangles A B C, A B′ C′, A B′ C, A B C′ compose the surface of the hemisphere A B C O, and the angles A, B, C—two right angles is the excess E; wherefore the preceding analogy becomes the sur. of 2 A B C : the sur. of A B C O :: the excess E : two right angles; or the sur. of A B C : the sur. of the sphere :: the excess E : eight right angles.

§ 3. OF RIGHT CONES AND CONIC FRUSTA DESCRIBED IN AND ABOUT SPHERES.

53. When a cone and conic frusta have their corresponding slant sides inscribed in the circumference of a great circle of a sphere, the *cone* and *conic frusta* are said to be *inscribed in the sphere;* and when a cone and conic frusta have their corresponding slant sides described about, or in contact with, the circumference of a great circle of a sphere, the *cone* and *conic frusta* are said to be *described about the sphere.*





Of the preceding solids, A B C is a right cone, and B C D E, D E F G are conic frusta, inscribed in the sphere of which the centre is O; and A′B′C′ is a right cone, and B′C′D′E′, D′E′F′G′ are conic frusta described about, or circumscribing, the sphere of which the centre is O′.

54. Two cones or conic frusta, which have their slant sides corresponding sides of regular and similar polygons, described in and about great circles of spheres, have evidently their bases and altitudes proportional, and are respectively similar. The comparison and determination of the surfaces and the volumes of these solids form the subject of this section.

THEOREM XVI.

55. *The curve surfaces of similar cones and conic frusta described in and about spheres, are proportional to the squares of the radii of the spheres.*

For the slant sides of the inscribed or circumscribed similar cones and conic frusta are proportional to the radii of the spheres; and the bases of those solids have also their radii and circumferences proportional to the radii of the spheres; wherefore the convex surfaces of the cones and conic frusta are respectively proportional to the squares of the radii of the spheres (22, 23, III).

THEOREM XVII.

56. *The curve surface of a cone or conic frustum described about a sphere, and having its slant sides bisected at the points of contact, is equal to the product of the altitude of the cone or frustum and the circumference of a great circle of the sphere.* (See fig. art. 53).

For if a small circle of the sphere be drawn parallel to the base of the circumscribed solid, at the middle of its slant side, or point of contact; and if a great circle be also drawn at this point, the radii and circumferences of these circles will evidently be proportional to the altitude and slant side of the solid; and the product of the slant side and the circumference of the small circle will therefore be equal to the product of the altitude and the circumference of the great circle (the. 5). But the first of these products is equal to the curve surface of the circumscribed solid (23, III); wherefore this surface is equal to the product of the altitude of the cone or conic frustum and the circumference of a great circle of the sphere.

Remark. It may be immediately inferred from articles 47, II, and 22, IV, that the base of a right cone and its curve surface are proportional to the corresponding radius and slant side; and this cone being described about a sphere, these right lines are also proportional to the radius of the sphere and the distance between its centre and the vertex of the cone. The following property belongs therefore to any right cone described about a sphere, and having its slant side bisected or not bisected at the point of contact, and may be considered as a theorem allied to the preceding:

The product of the curve surface of a cone, described about a sphere, and the radius of the sphere, is equal to the product of the base of the cone, and the distance between its vertex and the centre of the sphere.

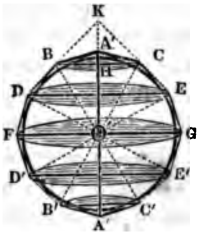
THEOREM XVIII.

57. *The volumes of similar cones and conic frusta described in and about spheres, are proportional to the cubes of the radii of the spheres.* (See fig. the. xx).

For the inscribed or circumscribed cones and conic frusta have their altitudes and the radii of their bases proportional to the radii of the spheres; wherefore their volumes are proportional to the cubes of the radii of the spheres (24, 26).

THEOREM XIX.

58. *The volume of a solid composed of cones and intermediate conic frustra circumscribing a sphere, is equal to one third of the product of its convex surface and the radius of the sphere.*



Let A A′ F G be a solid composed of the cones A B C, A′B′C′ and conic frustra B C D E, B′C′D′E′, &c. circumscribing the sphere of which the centre is O; the volume of A A′ F G is equal to one third of the product of its convex surface and the radius R of the sphere.

Join the circumferences B C, D E, &c. and the centre O by the successive conic surfaces B O C, D O E, &c. forming the partial conoidal solids A B C O, B C D E O, &c. which may be generated by the revolution of the isosceles triangles A B O, B D O, &c. about the common axis A A′. Let B H be the radius of the circle B C, and complete the cone D E K by producing the slant sides of the frustum D E B C. The conoidal solid A B C O is equal to the two cones A B C, O B C on the same base B C, and having their altitudes A H, O H together equal to O A. The volume of this solid is therefore equal to $\frac{1}{3}$ A O.circle B C (25). But the product O A.circle B C is equal to R.curve surface of the cone A B C (rem. 56); wherefore the volume of A B C O is equal to $\frac{1}{3}$ R.curve surface of A B C. For a like reason the volume of the solid K B C O is equal to $\frac{1}{3}$ R.curve surface of K B C, and the volume of the solid K D E O is equal to $\frac{1}{3}$ R.curve surface of K D E. Now the difference of these last solids is the solid B C D E O and the difference of their volumes is $\frac{1}{3}$ R.curve surface of the frustum B C D E; wherefore the volume of the conoidal solid B C D E O is $\frac{1}{3}$ R.curve surface B C D E; and similar expressions will evidently correspond to the volumes of the remaining conoidal solids, bounded by the curve surfaces D E F G, F G D′ E′, &c.; wherefore the volume of the compound solid A A′ F G is equal to one third of the product of the whole convex surface A B C D E, &c. and the radius R of the sphere.

Corollary. The volume of the compound solid A B C D E, described about the spheric segment A D E, together with the volume of the adjacent cone O D E, is equal to the volume of the conoidal solid A B C D E O, generated by the revolution of the polygonal sector A B D O about the axis A A′.

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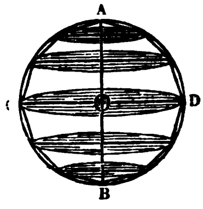
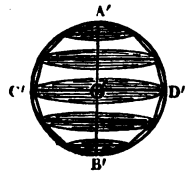
§ 4. OF THE AREAS OF THE SURFACES OF SPHERES.

THEOREM XX.

59. *The areas of the surfaces of spheres are proportional to the squares of their radii or diameters.*



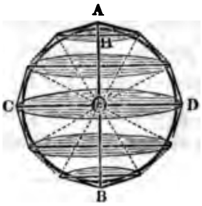


Let A B C D, A′B′C′D′ be spheres of which A O, A′O′ are the radii, the surface of A B C D : the surface of A′B′C′D′ :: $AO^2 : A'O'^2$.

For if similar solids composed of cones and conic frusta be inscribed in the spheres A B C D, A'B'C'D, their convex surfaces will be proportional to the squares of the radii A O, A'O' (54); and this proportionality will subsist whatever be the number of faces bounding the inscribed solids. But if this number be continually increased, the corresponding surfaces will also increase, and will continually approach the surfaces of A B C D, A'B'C'D', which form their respective limits, and which must have their invariable ratio $AO^2 : A'O'^2$. Wherefore the surface of A B C D : the surface of A'B'C'D' :: $AO^2 : A'O'^2$.

THEOREM XXI.

60. *The area of the surface of a sphere is equal to the product of the circumference of a great circle of the sphere and the diameter.*



Let A B C D be a sphere of which the diameter is A B; the surface of A B C D is equal to the product A B.circumference C D C.

For if a solid composed of cones and conic frusta be described about the sphere A B C D, the area of its surface will be equal to the product of its altitude and the circumference of a great circle of the sphere (55); and this area and corresponding product, by increasing the number of faces of the circumscribed solid, will continually approach the area of the surface of the sphere, and the product of the circumference of its great circle and the diameter A B, which form their respective limits, and between which there must subsist a corresponding equality. Wherefore the area of the surface of A B C D is equal to the product A B.circumference C D C.

Corollary. The area of the surface of a sphere is equal to four times the area of a great circle, and is also equal to the area of the curve surface of the circumscribing cylinder.

Remark. If a cone and consecutive conic frusta be described about any segment of a sphere, it may be proved by reasoning analogous to the preceding, that the area of the curve surface of the segment is equal to the product of its altitude and the circumference of a great circle of the sphere; so that the areas of the curve surfaces of different segments of the same sphere are respectively proportional to the altitudes of these segments.

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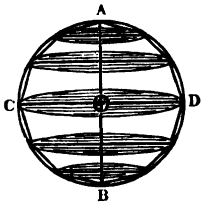
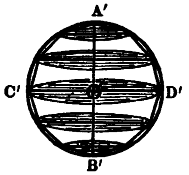
§ 5. OF THE VOLUMES OF SPHERES.

THEOREM XXII.

61. *The volumes of spheres are proportional to the cubes of their radii or diameters*





For if similar solids composed of cones and conic frusta be inscribed in the spheres A B C D, A′B′C′D′, their volumes will be proportional to the cubes of the radii A O, A′O′, or of the diameters A B, A′ B′, whatever be the number of their curve faces, and however near they may approach the spheres A B C D, A′B′C′D′, which form their respective limits, and which must have their invariable ratio $AO^3 : A'O'^3$. Wherefore the volume of A B C D : the volume of A′B′C′D′ :: $AO^3 : A'O'^3$.

THEOREM XXIII.

62. *The volume of a sphere is equal to one third of the product of the area of its surface and the radius.* (See fig. xxi).

Let A B C D be a sphere of which the radius is A O; the volume of A B C D is equal to one third of the product of its surface and the radius A O.

For if a solid composed of cones and conic frusta be described about the sphere A B C D, its volume will be equal to one third of the product of its convex surface and the radius A O; and this volume and corresponding product, by increasing the number of faces of the circumscribed solid, will continually approach the area of the sphere and one third of the product of its surface and the radius A O, which form their respective limits and between which there must subsist a corresponding equality. Wherefore the volume of A B C D is equal to one third the product of the surface of A B C D and the radius A O.

Corollary. The volume of a sphere is equal to two thirds of the volume of its circumscribing cylinder (12 and cor. 59).

Remark. The revolution of a sector of a great circle about its side or radius, will generate a conoidal solid which is termed a *spheric sector*, and of which the volume is proved as above to be one third the product of the corresponding spherical surface and the radius of the sphere. This solid is equal to a segment of the sphere and an adjacent cone; its volume and that of the cone being determined, the volume of the corresponding segment may be easily found.

CHAPTER IV.

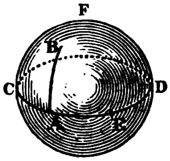
OF CONSTRUCTIONS AND MENSURATION RELATIVE TO CURVE SOLIDS.

63. THE problems resolved in this chapter complete practical geometry and the mensuration of elementary solids. The constructions in the first section are intentionally limited to the principal positions of circles on the surface of the sphere: the formulæ in the second section relate to the cylinder, cone, and sphere; and involve the determination of their curve surfaces and their corresponding total or partial volumes.

§ 1. OF CONSTRUCTIONS RELATIVE TO CIRCLES OF THE SPHERE.

PROBLEM I.

64. *Through a given point* A *or* B *on the surface of a sphere, to draw a perpendicular to a given great circle* C D.

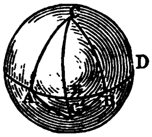


First, let the given point A be in the circle C D, and apply A E equal to a quadrant. From E as a centre describe through A the arc A B, which will evidently be perpendicular to the great circle C D.

Secondly, let the given point be B, from which at the distance of a quadrant, describe an arc intersecting C D in E. From the point E as a centre describe through B the arc A B, which will be perpendicular to C D.

PROBLEM II.

65. *To divide a given spherical arc* A B *into two equal parts.*



From A and B as centres, describe arcs intersecting in the points C, C′, from which at the distance of a quadrant determine the point D. Through C and C′ from D describe the arc C E C′ of a great circle, which will evidently bisect A B in the point E.

PROBLEM III.

66. *To divide a given spherical angle* A C B *into two equal parts.*

In the sides of the given angle C take equal parts C A, C B, and from the points A, B as centres describe arcs intersecting in C′. From C, C′ at the distance of a quadrant determine the point D, from which through C, C′ describe the bisecting arc C C′.

PROBLEM IV.

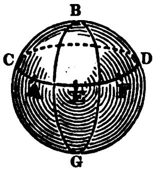
67. *To find the pole of any given circle* C D *of the sphere.* (See fig. art. 64, 68).

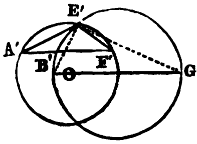
First, let C D be a great circle of the sphere, and from any two points A, E as centres, describe at the distance of a quadrant, arcs intersecting in F, which will be the pole of C D.

Secondly, let C D be a small circle, in which take equal arcs A E, E F. Bisect these arcs by great circles meeting in the points B, G (65), either of which will be the pole of the small circle C D.

PROBLEM V.

68. *To find the diameter drawn from a given point* B *in the surface of a sphere.*





From the given point B as a centre describe the circle C D, in which take any arcs A E, E F. Construct a plane triangle A′E′F′, of which the base is the chord of the arc A F and its sides the chords of A E, E F; and about the triangle describe the circle A′E′F′O. Through the centre O of the circle A′E′F′ draw the right line B′O G′; and from E′ with the distance E′B′, equal to the chord of the spherical arc B C, determine the point B′; join E′B′, perpendicular to which draw E′G′ meeting B′G′ in the point G′. The right line B′G′ is evidently the diameter of a circle passing through E′, and is also equal to the required diameter B G of the sphere.

For the circle A′E′F′ is equal to the circle C D of the sphere, and B′O is evidently equal to the perpendicular B O drawn from B to the plane of C D. The circle B′E′G′ is therefore equal to a great circle of the sphere, of which the diameter B G is equal to B′ G′.

As to the position of G on the surface of the sphere, it may be determined by a quadrant of the circle B′E′G′ and a great circle described from B, in accordance with the first case of the preceding problem.

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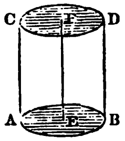
§ 2. OF MENSURATION RELATIVE TO CURVE SOLIDS.

PROBLEM I.

69. *To find the area of the curve surface of a right cylinder, of which the altitude and circumference of the base are given.*



The area$=a\,c$, ... [1],

in which a is the altitude and $c=2\,r\,\pi$ the circumference of the base.

Example. Let $a=10$ feet, and let the radius r of the base be 5. Then $c=31.416$, and the area is 314.16 square feet.

Remark. The total area of the cylinder is $c\,(a+r)$.

PROBLEM II.

70. *To find the volume of a cylinder of which the altitude and radius of the base are given.*

The volume$=a\,b$, ... [2],

in which a is the altitude and $b=r^2\pi$ the area of the base.

Example. Let $a=3$, $r=2$ yards. The volume is $12\,\pi=37.6992$ cubic yards.

PROBLEM III.

71. *To find the curve surface of a right cone of which the slant side and the circumference of the base are given.*

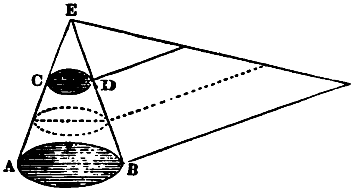
The area$=\frac{1}{2}\,c\,s$, ... [3], in which c is the circumference of the base and s is the slant side.

Example. Let $s=4$ feet, and let the radius r of the base be 2 feet 6 inches. Then $c=5\pi=15.708$, and the curve surface is 31.416 square feet.

Remark. The total area of the cone is $\frac{1}{2}c(r+s)$.

PROBLEM IV.

72. *To find the curve surface of the frustum of a right cone, of which the slant side and the circumferences of the bases are given.*



The area$=\frac{1}{2}s(c+c')$, ... [4],

in which s is the slant side and c, c' the given circumferences.

Example. Let $s=5$ yards, and let the radii r, r' of the bases be 3 and 2 yards. Then $\frac{1}{2}(c+c')=(r+r')\pi=15.708$ and the curve surface is 78.54 square yards.

Remark. The total area of the conic frustum is $\frac{1}{2}\{c(r+s)+c'(r'+s)\}$.

PROBLEM V.

73. *To find the volume of a cone of which the altitude and the radius of the base are given.*

The volume$=\frac{1}{3}ab$, ... [5],

which a is the altitude and $b=r^2\pi$ the area of the base.

Example. Let $a=6$ yards, and let the radius r of the base be 2 yards. Then $b=12.5664$ square yards, and the volume of the cone is 25.1328 cubic yards.

PROBLEM VI.

74. *To find the volume of the frustum of a cone of which the altitude and the areas of its bases are given.*

The volume$=\frac{1}{3}a(b+b'+\sqrt{bb'})$, ... [6]

in which a is the altitude and b, b' the areas of its base.

Example. Let $a=6$ feet, and let the radii r, r' of the lower and upper bases be 4 and 3 yards. Then $b=16\pi$, $b'=9\pi$, $\sqrt{bb'}=12\pi$, and the volume is $74\pi=232.4784$ cubic feet.

PROBLEM VII.

75. *To find the area of the surface of a sphere of which the radius is given.*

The area$=4r^2\pi$ . . . . [7],

in which r is the given radius.

Example. Let $r=2\frac{1}{2}$ yards. The area is $25\pi=78.54$ square yards.

PROBLEM VIII.

76. *To find the area of the curve surface of the segment of a sphere, of which the altitude aud radius of the surface are given.*

The area$=2ar\pi$, ... [8],

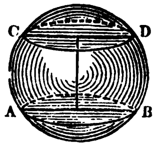
in which a is the altitude and r the radius of the surface.

Example. Let $r=12$ inches, $a=4$; the area is $96\pi=301.5936$ square inches.

Remark. The total area of the segment is $a\pi(4r-a)$.

PROBLEM IX.

77. *To find the area of a spheric zone, of which the altitude and radius of the surface are given.*



The area$=2ar\pi$, ... [9],

in which a is the altitude and r the radius of the surface.

For the zone is the difference of $4r^2\pi$ and $4r\pi(r-\frac{1}{2}a)$; wherefore its area is expressed by formula [9]. (See 75, 76).

Example. Let $a=6$ inches and $r=10$. The area is 376.992 square inches.

PROBLEM X.

78. *To find the volume of a sphere of which the radius is given.*

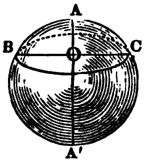
The volume$=\frac{4}{3}r^3\pi$, ... [10],

in which r is the given radius.

Example. Let $r=12$ inches. The volume is 4.1888 cubic feet.

PROBLEM XI.

79. *To find the volume of the segment of a sphere, of which the altitude and the radius of the surface are given.*



The volume$=\frac{1}{3}(3r-a)a^2\pi$, ... [11],

in which a is the altitude of the segment and r the radius of its curve surface.

For the segment is equal to the difference between a spheric sector and cone, the volumes of which are $\frac{2}{3}ar^2\pi$ and $\frac{1}{3}(2ar-a^2)(r-a)$, and their difference equal to the expression [11].

Example. Let $a=6$, $r=8$ inches. The volume of the segment is 678.5856 cubic inches.

Remark. Should the radius r' of the base of the segment be given, there will result $2ar=a^2+r'^2$, and thence the value of r; or the volume of the segment may in this case be determined from the expression $\frac{1}{6}(3r'^2+a^2)a\pi$, found from [11] by eliminating r. This expression is obviously equal to $\frac{1}{2}ar'^2\pi$ and $\frac{4}{3}(\frac{1}{2}a)^3\pi$, the first of which represents half a cylinder having the same base and altitude as the segment, and the second is the volume of a sphere corresponding to radius $\frac{1}{2}a$.

THE END.

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